Exercise 1 *Let (u_n) be the sequence defined by*

$$u_0 = 0,\ \forall\ n \in \mathbb{Z}_+,\ u_{n+1} = \frac{u_n + 2}{4 - u_n}.$$

Set $v_n = \frac{u_n - 1}{u_n - 2}$ and $f(x) = \frac{x+2}{4-x}$.

1. *Solve the equations $f(x) = 1$ and $f(x) = 2$.*
2. *Prove that if $u_n < 1$, then $u_{n+1} < 1$. Deduce that the sequence (u_n) is well defined.*
3. *Show that (v_n) is well defined and geometric. Deduce the expression of v_n with respect to n.*
4. *Derive the expression of u_n with respect to n. What is the limit of (u_n)?*

Exercise 2 *Determine the expression of u_n solution of*

$$\begin{cases} u_0 = 0,\ u_1 = 1, \\ u_{n+1} = -u_n - \frac{5}{36} u_{n-1},\ n \geq 1. \end{cases}$$

Exercise 3 *Write out the Cholesky algorithm. Determine the cost of the algorithm.*

Exercise 4 *Let A_α be the matrix*

$$A_\alpha = \begin{pmatrix} \alpha & 0 & -1 \\ 0 & \alpha & -1 \\ -1 & -1 & \alpha \end{pmatrix}$$

parametrized by $\alpha \in \mathbb{R}$.

1. *Determine the eigenvalues of A_α and some corresponding eigenvectors.*
2. *For which values of α the matrix A_α is invertible?*
3. (a) *For which values of α the matrix A_α admits a Cholesky decomposition?*
   (b) *For those values, compute the decomposition from the algorithm of Exercise 3.*
   (c) *By means of an up-down algorithm, solve explicitly the linear system $A_\alpha x = b$ for some $b \in \mathbb{R}^n$.*
   (d) *Deduce the expression of A_α^{-1}.*
   (e) *For a symmetric positive-definite matrix, the condition number of a matrix is the ratio of the largest eigenvalue by the lowest one. Compute the condition number of A_α.*
   (f) *Which asymptotics upon α corresponds to the most suitable case for inverting A_α?*
4. *We assume in this question that $\alpha \neq 0$. We aim at approximating the solution of $A_\alpha x = b$ by means of an iterative method, which is the Jacobi method: writing $A_\alpha = D_\alpha - E - F$ where D_α is the diagonal matrix whose coefficients are the diagonal entries of A_α, E is lower triangular and F is upper triangular. The Jacobi method consists of the sequence $D_\alpha x^{n+1} = (E+F)x^n + b$.*
   (a) *Write down the matrices D_α, E and F.*
   (b) *For which values of α is the method convergent?*
   (c) *What are the induction relations for x_1^n, x_2^n and x_3^n.*
   (d) *Compute the expression of x^n with respect to n.*
   (e) *What is the limit of x^n as n goes to ∞? Comment.*

Exercise 5 *Give the solutions to the following ODEs:*

$$\hat{y}'(t) = 2\hat{y}(t) + 1; \qquad \hat{y}'(t) = \frac{t}{t^2+1}\hat{y}(t); \qquad \hat{y}'(t) = -(\tan t)\hat{y}(t) + \cos t; \qquad t\hat{y}'(t) = -\hat{y}(t) + t,$$

supplemented with the initial condition $\hat{y}(t_0) = y_0$. *Specify for which* t_0 *such solutions exist.*

Exercise 6

1. *Determine the solutions to the second-order ODE*

$$y''(t) - 2y'(t) + y(t) = 0.$$

2. *We then focus on the ODE*

$$\begin{cases} y''(t) - 2y'(t) + y(t) = \cos t, \\ y(0) = 0, \\ y'(0) = 1. \end{cases} \tag{1}$$

   (a) *Prove that there exists a unique solution to* (1).

   (b) *What are the eigenvalues of the corresponding matrix?*

   (c) *Determine C such that* $y(t) = C(t)e^t$ *satisfies* (1).

   (d) *Conclude.*

Exercise 7 *We consider the ordinary differential equation*

$$(t^2+1)\hat{y}'(t) + t = t\hat{y}(t)^2. \tag{2}$$

1. *Prove the existence of a solution to* (2) *defined for* $t \in \mathbb{R}$.
2. *Determine a constant solution to* (2).
3. *Set* $\hat{z} = \hat{y} - 1$. *Show that* $\hat{z}$ *satisfies*

$$(t^2+1)\hat{z}'(t) = t\big(2\hat{z}(t) + \hat{z}^2(t)\big). \tag{3}$$

4. *Set* $\hat{w} = \frac{1}{\hat{z}}$. *Determine the equation satisfied by* $\hat{w}$ *and solve* (2).
5. *Apply the explicit and implicit Euler schemes to* (2).

Exercise 8 *These equations model the evolution of an isolated predator-prey system (for instance rabbits and lynx):*

$$\begin{cases} x'(t) = x(t)\big(3 - y(t)\big), & x(0) = 1, \\ y'(t) = y(t)\big(x(t) - 2\big), & y(0) = 2. \end{cases} \tag{4}$$

1. *Determine which variable corresponds to the number of preys.*
2. *Show that there is no constant solution to Problem* (4).
3. *Rewrite Eqs.* (4) *as* $\mathbf{Y}'(t) = \mathbf{F}\big(\mathbf{Y}(t)\big)$, *where* $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$. *Deduce that there exists a unique maximal solution* $\mathbf{Y}$.
4. *Prove that x and y cannot vanish. Deduce the sign of each unknown.*
5. *We set* $H(x,y) = x - 2\ln x + y - 3\ln y$. *H is called the* Hamiltonian *of the system. Show that for all* $t \geq 0$, $H\big(x(t), y(t)\big) = H\big(x(0), y(0)\big)$.

6. *Apply the explicit Euler scheme to Eq.* (4). *Is the numerical Hamiltonian also constant?*
7. *We introduce the symplectic Euler scheme:*

$$\begin{cases} x_{n+1} = x_n + \Delta t x_{n+1}(3 - y_n), \\ y_{n+1} = y_n + \Delta t y_n(x_{n+1} - 2). \end{cases}$$

   Prove this scheme is consistant.

In the sequel, $f : [0,1] \times \mathbb{R} \longmapsto \mathbb{R}$ denotes a smooth function of class $\mathscr{C}^1$. We aim at approximating the solution of the ODE

$$\hat{y}'(t) = f\big(t, \hat{y}(t)\big). \tag{5}$$

Let T be some positive number and $N \in \mathbb{Z}_+$, $N \neq 0$. Then we set $\Delta t = \frac{T}{N}$ and $t^n = n\Delta t$, $0 \leq n \leq N$.

Exercise 9 *We assume in this exercise that $f(t, y) = -y$.*

1. *Solve* (5) *in this case supplemented with the initial condition $\hat{y}(0) = 1$.*
2. *Apply the explicit Euler scheme to construct the sequence (y_n).*
3. *Yield the explicit expression of y_n with respect to n and Δt. Is the scheme relevant for any Δt?*
4. *Compare y_N and $\hat{y}(T)$. Conclude.*
5. *Follow the same directions about the Heun scheme.*
6. *Which scheme seems to be the most efficient?*

Exercise 10 *We take in this exercise $f(t, y) = 1 - 2y$.*

1. *Solve* (5) *in this case supplemented with the initial condition $\hat{y}(0) = 1$.*
2. *Apply the implicit Euler scheme to construct the sequence (y_n).*
3. *Yield the explicit expression of y_n with respect to n and Δt.*

Exercise 11 *To provide an approximate solution to* (5)*, we propose the scheme*

$$\frac{3y_{n+2} - 4y_{n+1} + y_n}{2} = \Delta t\, f(t^{n+2}, y_{n+2}).$$

1. *How can this scheme be initialized?*
2. *Show that the scheme is convergent. Determine its order.*
3. *Is this scheme explicit?*
4. *In the case $f(t, y) = -y$, solve the linear inductive relation for y_n.*
5. *Propose a modification of the right hand side in the previous scheme to improve the order.*

Exercise 12 *The enhanced Euler scheme reads*

$$y_{n+1} = y_n + \Delta t f\left(t^n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} f(t^n, y_n)\right).$$

1. *Compute* $\hat{y}''(t)$ *for* $\hat{y}$ *solution of* (5).
2. *Show this scheme is convergent and determine its order.*

Exercise 13 *We aim at studying a numerical scheme dedicated to the resolution of the autonomous ordinary differential equation*

$$\begin{cases} y'(t) = f\big(y(t)\big), & \text{(6a)} \\ y(0) = y_0, & \text{(6b)} \end{cases}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is a* $\mathscr{C}^\infty$ *function. We consider the* lintrap *scheme*

$$\frac{y_{n+1} - y_n}{\Delta t} = f(y_n) + f'(y_n)\frac{y_{n+1} - y_n}{2} \tag{7}$$

where $\Delta t > 0$ *is some positive number and we set* $t^n = n\Delta t$, $n \geq 0$.

1. *Does there exist a unique solution to ODE* (6)*?*
2. ***Study of the numerical scheme***
   (a) *Is the scheme explicit or implicit?*
   (b) *Is the scheme well-defined in any case? Show that if* f *is monotone-decreasing, then the scheme is well-defined.*
   (c) *For* $\hat{y}$ *solution to* (6), *compute* $\hat{y}''(t)$.
   (d) *Prove that* (7) *is a consistant scheme up to order* 2.
3. ***Investigation of a particular case.*** *We suppose in this question that* $f(y) = (y+1)^2$.
   (a) *Compute the exact solution* $\hat{y}$ *to* (6) *in that case.*
   (b) *Apply Scheme* (7) *to ODE* (6). *Express* y_{n+1} *as a function of* Δt *and* y_n.
   (c) *Let us introduce* $z_n = \frac{1}{y_n+1}$. *Show that* (z_n) *satifies an arithmetic progression.*
   (d) *Deduce the expression of* y_n *with respect to* n *and* Δt.
   (e) *Compare* y_n *and* $\hat{y}(t^n)$. *Conclude.*

Exercise 14 *Let us consider the system*

$$\begin{cases} x'(t) = y(t),\ x(0) = 1, \\ y'(t) = -x(t),\ y(0) = 0. \end{cases} \tag{8}$$

1. *Prove that the trajectories* $t \longmapsto \big(x(t), y(t)\big)$ *are included in the unit circle* $x^2 + y^2 = 1$.
2. *Write out the explicit Euler scheme, the implicit Euler scheme and the Crank-Nicholson scheme for the resolution of* (8).
3. *Do these schemes preserve the trajectories?*

Exercise 15 *Let us consider the ODE*

$$\begin{cases} \hat{y}'(t) + 3\hat{y}(t)^2 = 0, \\ \hat{y}(0) = 1. \end{cases} \tag{9}$$

1. *Solve* (9)*. What is the limit of* $\hat{y}(t)$ *as* $t \to +\infty$*?*
2. *Apply the explicit Euler scheme to this ODE and study the limit as* $n \to +\infty$.
3. *We propose the following scheme*

$$\frac{y_{n+1} - y_n}{\Delta t} + 3\left(\frac{y_{n+1} + y_n}{2}\right)^2 = 0. \tag{10}$$

   (a) *Derive an expression of* y_{n+1} *with respect to* y_n *and deduce a condition upon* Δt *for the limit to be correct as* $n \to +\infty$.

   (b) *Show this scheme is consistant at order* 2.

Exercise 16 *Let us consider the following differential equation*

$$-u''(x) + u'(x) + \left(\alpha^2 - \frac{1}{4}\right)u(x) = f(x), \tag{11}$$

where $\alpha > 0$ *is some real number and* f *is a continuous function over* $\mathbb{R}_+$*. To supplement Equation* (11)*, we propose two types of boundary conditions:*

$$u(0) = 0, \qquad u'(0) = 1, \tag{BC1}$$

$$u(0) = 0, \qquad u(1) = 2. \tag{BC2}$$

1. ***We assume in this question ONLY that*** $f(x) = 0$ ***for all*** $x \geq 0$.

   (a) *Compute the expression of the solution to* (11) *together with* (BC1).

   (b) *What is the solution for* (BC2)*?*

2. ***General case.***

   (a) *Prove that there exists a unique solution to Equation* (11) *(for some given* f*) supplemented with* (BC1).

   (b) *What can we say about the problem* (11)–(BC2)*?*

   (c) *Let us set*

$$\forall\, x \geq 0,\ \hat{u}(x) = e^{(\frac{1}{2}+\alpha)x}\left(c_0 - \frac{1}{2\alpha}\int_0^x f(y)e^{-(\frac{1}{2}+\alpha)y}\,\mathrm{d}y\right) + e^{(\frac{1}{2}-\alpha)x}\left(d_0 + \frac{1}{2\alpha}\int_0^x f(y)e^{-(\frac{1}{2}-\alpha)y}\,\mathrm{d}y\right).$$

   Show that $\hat{u}$ *satisfies* (11).

   (d) *Determine* (c_0, d_0) *so that* $\hat{u}$ *also satisfies* (BC1)*. Same question for* (BC2).

   (e) *Is this expression for* $\hat{u}$ *always useful?*

3. ***Numerical approach.*** *Let us set* $\Delta x = \frac{1}{N-1}$ *for some integer* $N \geq 2$ *and* $x_i = (i-1)\Delta x$ *for* $i \in \{1, \ldots, N\}$*. In this section, we are interested in designing a numerical scheme to provide approximations* u_i *of* $\hat{u}(x_i)$.

   (a) *Propose a finite-difference scheme to approximate the solution to* (11).

   (b) *How to take* (BC1) *into account? Write out the corresponding algorithm to compute* u_i *for all* $i \in \{1, \ldots, N\}$.

(c) *Same question for* (BC2). *What can you say about the matrix of the underlying linear system?*

4. ***Substitution.*** *Let v be the function such that $u(x) = v(x)e^{x/2}$.*

   (a) *Prove that v is a solution of the following equation*

$$-v''(x) + \alpha^2 v(x) = f(x)e^{-x/2}. \tag{12}$$

   (b) *What are the boundary conditions for v corresponding to* (BC2)*?*

   (c) *Propose a finite-difference numerical scheme to solve* (12)–(BC2). *Does this seem more practical than in Question 3.(c)?*

   (d) *We admit that the Cholesky factorization of a tridiagonal matrix $A \in \mathscr{M}_n(\mathbb{R})$ is $B^T B$ where B is a bidiagonal upper matrix of the form*

$$\begin{pmatrix} \sqrt{\beta_1} & \gamma_2 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \gamma_{n-1} \\ 0 & \cdots & 0 & \sqrt{\beta_n} \end{pmatrix}$$

   Adapt the Cholesky algorithm to the factorization of the matrix of Question 4.(c). In particular, show that (β_i) satisfies the inductive relation

$$\beta_i + \frac{1}{\beta_{i-1}} = 2 + \alpha^2 \Delta x^2, \qquad \beta_1 = 2 + \alpha^2 \Delta x^2.$$

   What is γ_i equal to?

Exercise 17 *We aim at solving the autonomous (*i.e. *f does not depend on t) ordinary differential equation*

$$\begin{cases} y'(t) = f\big(y(t)\big), \\ y(0) = \frac{1}{2}. \end{cases} \tag{13}$$

We assume that f is of class $\mathscr{C}^2(\mathbb{R})$.

1. *Justify that ODE* (13) *has a unique equation denoted $\hat{y}$. What is the regularity of $\hat{y}$?*

2. *Give the expression of the solution in the following cases:*

   (a) *$f(x) = 1$;*

   (b) *$f(x) = \lambda x$ with $\lambda \in \mathbb{R}$.*

3. *In the general case, we cannot provide an explicit expression. That is why we aim at constructing approximate values of the solution at some points. More precisely, we set*

$$t^n = n\Delta t,\ \Delta t = \frac{3}{N},$$

   for some fixed integer $N \geq 1$. A numerical scheme is a method whose purpose is to compute y_n which is an approximation of $\hat{y}(t^n)$.

   (a) *Do we have $y_n = \hat{y}(t^n)$?*

   (b) i. *Apply the explicit Euler scheme to ODE* (13). *Express y_{n+1} as a function of y_n.*

      ii. *How many values do we need in order to initialize the sequence (y_n)?*

iii. *Write out the algorithm leading to the computation of the sequence* (y_n).

iv. *Recall the order of this scheme.*

(c) *We are interesting in the multi-step scheme*

$$z_{n+3} - z_{n+1} = \Delta t\left(\frac{7}{3}f(z_{n+2}) - \frac{2}{3}f(z_{n+1}) + \frac{1}{3}f(z_n)\right). \tag{14}$$

i. *Prove the consistency of Scheme* (14).

ii. *Study its stability.*

iii. *Deduce that this scheme is convergent.*

iv. *Is this scheme explicit or implicit? Justify your answer.*

v. *Determine the order of Scheme* (14).

vi. *How many values do we need in order to initialize the sequence* (z_n)*? Explain how to compute these initiliazing values.*

vii. *Write out the algorithm leading to the computation of the sequence* (z_n).

viii. *Which scheme would you recommend: Euler (Q. 3.(b)) or Scheme* (14)*?*

ix. *Apply Scheme* (14) *when* $f(x) = 1$. *Compute the exact expression of* z_n *for all* n.

Exercise 18 *We focus in this exercise on the ordinary differential equation*

$$\begin{cases} x_1'(t) = -x_1(t) - x_1(t)x_2(t), & x_1(0) = \dfrac{1}{3}, \\ x_2'(t) = -\dfrac{x_2(t)}{x_1(t)}, & x_2(0) = \dfrac{2}{3}. \end{cases} \tag{15}$$

We set $F(x,y) = \begin{pmatrix} -x - xy \\ -\frac{y}{x} \end{pmatrix}$ *and* $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

1. *Rewrite ODE* (15) *by means of* X *and* F.

2. *Does there exist a solution to ODE* (15)*?*

3. *Is it possible to provide an explicit expression of the solution?*

4. *Let* $\Delta t > 0$ *be such that* $\Delta t < \dfrac{1}{2}$. *We propose the numerical scheme*

$$\begin{cases} \dfrac{x_1^{n+1} - x_1^n}{\Delta t} = -x_1^n - x_1^n x_2^{n+1}, & x_1^0 = \dfrac{1}{3}, \\ \dfrac{x_2^{n+1} - x_2^n}{\Delta t} = -\dfrac{x_2^{n+1}}{x_1^n}, & x_2^0 = \dfrac{2}{3}. \end{cases} \tag{16}$$

   (a) *Show by induction that the sequences* (x_1^n) *and* (x_2^n) *belong to* $(0,1)$ *and are monotone-decreasing.*

   (b) *Deduce that they are convergent. Determine their limits.*

   (c) *Is Scheme* (16) *explicit?*

   (d) *Express* x_1^{n+1} *and* x_2^{n+1} *as functions of* x_1^n, x_2^n *and* Δt. *Deduce that this one-step scheme is consistant.*

Exercise 19 *We now study the pure advection equation*

$$\begin{cases} \partial_t Y(t,x) + \alpha \partial_x Y(t,x) = 0, & x \in (0,1), \\ Y(t,0) = 0, \\ Y(0,x) = Y_0(x), \end{cases} \tag{17}$$

with the same assumptions for α *and* Y_0 *as in the previous exercise.*

1. *Let Y_1 and Y_2 be two smooth solutions of PDE* (17). *Show that $Y_1 = Y_2$ by using*

$$E(t) = \int_0^1 |Y_1(t,x) - Y_2(t,x)|^2 \, \mathrm{d}x.$$

2. *Deduce that function $\hat{Y}$ defined by*

$$\hat{Y}(t,x) = \begin{cases} Y_0(x-\alpha t), & \text{if } x - \alpha t \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

is the unique smooth solution of (17).

3. *We now aim at studying numerical schemes simulating PDE* (17). *To do so, let us introduce $N_x \geq 2$ an integer and $\Delta t > 0$ a real number. We then set*

$$\Delta x = \frac{1}{N_x - 1}, \; x_i = (i-1)\Delta x, \; 1 \leq i \leq N_x, \qquad \text{and} \qquad t^n = (n-1)\Delta t, \; n \geq 1.$$

We propose the following schemes

$$\frac{Y_i^{n+1} - Y_i^n}{\Delta t} + \alpha \frac{Y_i^n - Y_{i-1}^n}{\Delta x} = 0, \tag{18}$$

$$\frac{Y_i^{n+1} - Y_i^n}{\Delta t} + \alpha \frac{Y_i^{n+1} - Y_{i-1}^{n+1}}{\Delta x} = 0. \tag{19}$$

   (a) ***Upwind scheme*** (18)*: write out the algorithm corresponding to the computation of Y_i^n for all $i \in \{1, \ldots, N_x\}$ and $n \geq 1$.*

   (b) ***Upwind scheme*** (18)*: show that this scheme is consistant and using Exercise 20, derive a stability condition.*

   (c) ***Implicit scheme*** (19)*: write out the corresponding algorithm. Does it require the resolution of a linear system?*

Exercise 20 *Let us study the 1D advection–diffusion equation*

$$\begin{cases} \partial_t Y(t,x) + \alpha \partial_x Y(t,x) - \nu \partial_{xx}^2 Y(t,x) = 0, & x \in (0,1), \\ Y(t,0) = Y(t,1) = 0, \\ Y(0,x) = Y_0(x), \end{cases} \tag{20}$$

for some constant velocity field $\alpha > 0$ and constant diffusion coefficient $\nu \neq 0$. The initial datum Y_0 is assumed to be smooth.

1. *Propose a discretization of* (20) *inspired by the previous implicit scheme. Does it require the resolution of a linear system? If so, what can you say about the matrix?*

2. *Let us set $Z(t,x) = Y(t,x) \exp\left[\frac{-\alpha}{4\nu}(2x - \alpha t)\right]$. Show that Z satisfies the following PDE*

$$\partial_t Z - \nu \partial_{xx}^2 Z = 0 \tag{21}$$

with suitable initial and boundary conditions. Do you think it is more relevant to discretize the equivalent equation (21)*?*