Exam

No document is authorized except the lecture notes with no inscription inside. Cell phones and computers are forbidden.

The final exam has two pages and consists of two independent exercises.

Exercise 1 *We aim at studying a numerical scheme dedicated to the resolution of the autonomous ordinary differential equation*

$$\begin{cases} y'(t) = f\big(y(t)\big), & \text{(1a)} \\ y(0) = y_0, & \text{(1b)} \end{cases}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is a* $\mathscr{C}^\infty$ *function. We consider the* lintrap *scheme*

$$\frac{y_{n+1} - y_n}{\Delta t} = f(y_n) + f'(y_n)\frac{y_{n+1} - y_n}{2} \tag{2}$$

where $\Delta t > 0$ *is some positive number and we set* $t^n = n\Delta t$, $n \geq 0$.

1. *Does there exist a unique solution to ODE* (1)*?*
2. ***Study of the numerical scheme***
   - (a) *Is the scheme explicit or implicit?*
   - (b) *Is the scheme well-defined in any case? Show that if* f *is monotone-decreasing, then the scheme is well-defined.*
   - (c) *For* $\hat{y}$ *solution to* (1), *compute* $\hat{y}''(t)$.
   - (d) *Prove that* (2) *is a consistant scheme up to order* 2.
3. ***Investigation of a particular case.*** *We suppose in this question that* $f(y) = (y+1)^2$.
   - (a) *Compute the exact solution* $\hat{y}$ *to* (1) *in that case.*
   - (b) *Apply Scheme* (2) *to ODE* (1). *Express* y_{n+1} *as a function of* Δt *and* y_n.
   - (c) *Let us introduce* $z_n = \frac{1}{y_n+1}$. *Show that* (z_n) *satifies an arithmetic progression.*
   - (d) *Deduce the expression of* y_n *with respect to* n *and* Δt.
   - (e) *Compare* y_n *and* $\hat{y}(t^n)$. *Conclude.*

Exercise 2 *Let us consider the following differential equation*

$$-u''(x) + u'(x) + \left(\alpha^2 - \frac{1}{4}\right)u(x) = f(x), \tag{3}$$

where $\alpha > 0$ *is some real number and* f *is a continuous function over* $\mathbb{R}_+$. *To supplement Equation* (3), *we propose two types of boundary conditions:*

$$u(0) = 0, \qquad u'(0) = 1, \tag{BC1}$$

$$u(0) = 0, \qquad u(1) = 2. \tag{BC2}$$

1. ***We assume in this question ONLY that*** $f(x) = 0$ ***for all*** $x \geq 0$.
   - (a) *Compute the expression of the solution to* (3) *together with* (BC1).
   - (b) *What is the solution for* (BC2)*?*

2. ***General case.***

   (a) *Prove that there exists a unique solution to Equation* (3) *(for some given f) supplemented with* (BC1).

   (b) *What can we say about the problem* (3)–(BC2)*?*

   (c) *Let us set*

$$\forall\, x \geq 0,\ \hat{u}(x) = e^{(\frac{1}{2}+\alpha)x}\left(c_0 - \frac{1}{2\alpha}\int_0^x f(y)e^{-(\frac{1}{2}+\alpha)y}\,\mathrm{d}y\right) + e^{(\frac{1}{2}-\alpha)x}\left(d_0 + \frac{1}{2\alpha}\int_0^x f(y)e^{-(\frac{1}{2}-\alpha)y}\,\mathrm{d}y\right).$$

   Show that $\hat{u}$ satisfies (3).

   (d) *Determine* (c_0, d_0) *so that $\hat{u}$ also satisfies* (BC1). *Same question for* (BC2).

   (e) *Is this expression for $\hat{u}$ always useful?*

3. ***Numerical approach.*** *Let us set* $\Delta x = \frac{1}{N-1}$ *for some integer* $N \geq 2$ *and* $x_i = (i-1)\Delta x$ *for* $i \in \{1, \ldots, N\}$. *In this section, we are interested in designing a numerical scheme to provide approximations u_i of* $\hat{u}(x_i)$.

   (a) *Propose a finite-difference scheme to approximate the solution to* (3).

   (b) *How to take* (BC1) *into account? Write out the corresponding algorithm to compute u_i for all* $i \in \{1, \ldots, N\}$.

   (c) *Same question for* (BC2). *What can you say about the matrix of the underlying linear system?*

4. ***Substitution.*** *Let v be the function such that* $u(x) = v(x)e^{x/2}$.

   (a) *Prove that v is a solution of the following equation*

$$-v''(x) + \alpha^2 v(x) = f(x)e^{-x/2}. \tag{4}$$

   (b) *What are the boundary conditions for v corresponding to* (BC2)*?*

   (c) *Propose a finite-difference numerical scheme to solve* (4)–(BC2). *Does this seem more practical than in Question 3.(c)?*

5. ***Bonus.*** *Careful: this question is apart from the grading process. It corresponds to extra points.*

   (a) *Write out the Cholesky algorithm.*

   (b) *We admit that the Cholesky factorization of a tridiagonal matrix* $A \in \mathscr{M}_n(\mathbb{R})$ *is* $B^T B$ *where B is a bidiagonal upper matrix of the form*

$$\begin{pmatrix} \sqrt{\beta_1} & \gamma_2 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \gamma_{n-1} \\ 0 & \cdots & 0 & \sqrt{\beta_n} \end{pmatrix}$$

   Adapt the Cholesky algorithm to the factorization of the matrix of Question 4.(c). In particular, show that (β_i) satisfies the inductive relation

$$\beta_i + \frac{1}{\beta_{i-1}} = 2 + \alpha^2\Delta x^2, \qquad \beta_1 = 2 + \alpha^2\Delta x^2.$$

   What is γ_i equal to?