Exercises #1

Exercise 1 *For $n \in \mathbb{Z}_+$, set $I_n = \displaystyle\int_0^{+\infty} t^n e^{-t}\,\mathrm{d}t$ and $J_n = \displaystyle\int_0^{+\infty} t^n e^{-t^2}\,\mathrm{d}t$.*

1. *By means of an integration by parts, derive a relation between I_n and I_{n-1}. Compute I_0 and then I_n for all n.*
2. *Same questions for J_n knowing that $J_0 = \frac{\sqrt{\pi}}{2}$.*

Exercise 2 *Let (u_n) be the sequence defined by*

$$u_0 = 0,\ \forall\, n \in \mathbb{Z}_+,\ u_{n+1} = \frac{u_n + 2}{4 - u_n}.$$

Set $v_n = \frac{u_n - 1}{u_n - 2}$ and $f(x) = \frac{x+2}{4-x}$.

1. *Solve the equations $f(x) = 1$ and $f(x) = 2$.*
2. *Prove that if $u_n < 1$, then $u_{n+1} < 1$. Deduce that the sequence (u_n) is well defined.*
3. *Show that (v_n) is well defined and geometric. Deduce the expression of v_n with respect to n.*
4. *Derive the expression of u_n with respect to n. What is the limit of (u_n)?*

Exercise 3 *Determine the expression of u_n solution of*

$$\begin{cases} u_0 = 0,\ u_1 = 1, \\ u_{n+1} = -u_n - \frac{5}{36} u_{n-1},\ n \geq 1. \end{cases}$$

Exercise 4 *Write out the Cholesky algorithm.*

Exercise 5 *Let A be the matrix*

$$A_\alpha = \begin{pmatrix} \alpha & -1 & 0 \\ -1 & \alpha & -1 \\ 0 & -1 & \alpha \end{pmatrix}$$

parametrized by $\alpha \in \mathbb{R}$.

1. *Compute the eigenvalues of A_α and some corresponding eigenvectors.*
2. *For which values of α the matrix A_α is invertible?*
3. *For which values of α the matrix A_α admits a Cholesky decomposition? For those values, compute the decomposition.*
4. *We assume $\alpha \neq 0$. Apply the Jacobi method which is the iterative method with $M = diag(A_\alpha)$ and $N = diag(A_\alpha) - A_\alpha$. For which values of α the method is convergent?*
5. *Solve explicitly $A_\alpha x = b$ for some $b \in \mathbb{R}^n$ and α inferred at Question 2.*

Exercise 6 *Let A be the matrix*

$$A = \begin{pmatrix} 3 & -2 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -2 \\ 0 & \cdots & 0 & -1 & 3 \end{pmatrix}.$$

Find out the LU decomposition of A. Hint: show first that L and U are bidiagonal. Then set $u_i = U_{ii}$, $v_i = U_{i,i+1}$, $\ell_i = L_{i+1,i}$ and $w_i = \frac{u_i - 1}{u_i - 2}$. Prove that (w_i) is geometric.