PW #3: transport equation

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The aim of this project is to compare numerical schemes designed to solve the linear transport equation:

$$\begin{cases} \partial_t Y(t,x) + u(t,x)\cdot\partial_x Y(t,x) = f(t,x), & \text{(1a)} \\ Y(0,x) = Y_0(x), & \text{(1b)} \\ Y(t,0) = \alpha, & \text{(1c)} \end{cases}$$

in the bounded domain $[0,1]$. The velocity field $u \geq 0$, the source term f, the initial condition Y_0 and the boundary datum $\alpha \in \mathbb{R}$ are **given**.

The domain is discretized by means of the nodes $x_i = (i-1)\Delta x$, with $\Delta x = \frac{1}{N+1}$ and $1 \leq i \leq N$ for some $N \in \mathbb{N}^*$. As for the time, we set $t^n = n\Delta t$ for some $\Delta t > 0$.

Exercise 1 (*Upwind method for constant velocity*)

*We set **in this exercise** $u(t,x) = u_0 = 2$, $f(t,x) = 0$, $Y_0(x) = \cos(2\pi x)$ and $\alpha = 1$. We focus on the **upwind method** which reads:*

$$\frac{Y_i^{n+1} - Y_i^n}{\Delta t} + u_0 \frac{Y_i^n - Y_{i-1}^n}{\Delta x} = 0.$$

1. *Implement this scheme.*
2. *Given a number of intervals N, set $\Delta t = \Delta x/2$. How many iterations are necessary to reach the steady state?*
3. *What happens when setting $\Delta t = \Delta x$?*
4. *For some N and $\Delta t = \Delta x/2$, compute the error with respect to time. What do you observe?*
5. *For $\Delta t = \Delta x/2$, compute the error at time $t = 0.25$ for $N = 10^2, 10^3, 10^4, 10^5$. What do you conclude about this scheme?*

Exercise 2 (*Upwind method in the periodic case*)

We replace the boundary condition (1c) by the periodic assumption:

$$Y(t,0) = Y(t,1). \tag{2}$$

*We then set **in this exercise** $u(t,x) = u_0 = 2$ and:*

$$Y_0(x) = \begin{cases} 0, & \text{if } x \in (0,0.3) \cup (0.7,1), \\ 1, & \text{if } x \in (0.4,0.6), \\ 10x-3, & \text{if } x \in (0.3,0.4), \\ 7-10x, & \text{if } x \in (0.6,0.7). \end{cases}$$

We plan to solve this problem with the same scheme as in Ex. 1.

1. *Case* $f(t,x) = 0$.
   (a) *For* $\Delta t = \Delta x/2$*, compute the numerical solution. What do you observe as time goes by?*
   (b) *The exact solution at time* $\frac{k}{2}$, $k \in \mathbb{N}$, *is* $Y(\frac{k}{2}, x) = Y_0(x)$. *For* $k = 1,2,3,4,5$*, plot the error. Interpret this result.*
   (c) *Does the numerical solution remain bounded? What are these bounds?*

2. *Case* $f(t,x) = \cos(10\pi t)$.
   (a) *Does the conclusion of Q. 1(c) still hold?*
   (b) *Plot the error at time* $t = 1$ *for* Δx *decreasing.*

Exercise 3 (*An unstable scheme*)

We take data as in Ex. 1. Implement the downwind scheme:

$$\frac{Y_i^{n+1} - Y_i^n}{\Delta t} + u_0 \frac{Y_{i+1}^n - Y_{i-1}^n}{2\Delta x} = 0.$$

1. *Is this scheme consistent with Eq.* (1a)*?*
2. *Using numerical simulations, show that this scheme is unconditionally unstable.*

We are interested in another numerical method devoted to advection operators as in (1). It is described by the following algorithm:

Algorithm 1 Method of characteristics

while $n < n_{max}$ **do**
  for i from 1 to N **do**
    Compute $\xi \leftarrow x_i - \Delta t \cdot u_i^n$
    if $\xi \leq 0$ **then**
      $Y_i^{n+1} \leftarrow \alpha$
    else
      Find j such that $\xi \in [x_j, x_{j+1})$
      Set $\theta \leftarrow (x_{j+1} - \xi)/\Delta x$
      $Y_i^{n+1} \leftarrow \theta Y_j^n + (1-\theta) Y_{j+1}^n$
    end if
  end for
  $n \leftarrow n+1$
end while

Exercise 4 (*Two numerical schemes for a non-trivial case*)

We set $u(t,x) = tx(1-x)$, $f(t,x) = 0$, $\alpha = 0$ *and* Y_0 *as in Ex. 2.*

1. *Implement the method of characteristics (MOC).*
2. *Show that this scheme is unconditionally stable,* i.e. *that the time step* Δt *is not necessarily related to the mesh size* Δx.
3. *Determine at a fixed time* t^n *the maximal value* $\mathscr{U}^n$ *of* $x \mapsto u(t^n, x)$.
4. *Set* $\Delta t^n = \Delta x / \mathscr{U}^n$. *Compare at each iteration the corresponding solutions to the upwind scheme and to MOC. Which scheme seems to be the most suitable one?*