Project #1: numerical simulations of discontinuities

This project is divided into two parts: the first one deals with the design of an antidiffusive scheme to solve the linear transport equation, while the second one is devoted to the numerical resolution of the 1D Euler equations. It is mainly inspired from the paper *Contact Discontinuity Capturing Schemes for Linear Advection and Compressible Gas Dynamics* written by B. Després and F. Lagoutière (2001).

1 Transport equation with discontinuous initial data

We aim at solving the following problem in dimension 1:

$$\begin{cases} \partial_t \beta + u \partial_x \beta = 0, \\ \beta(0, \cdot) = \beta_0, \end{cases} \tag{1}$$

for a constant velocity field $u > 0$ and a bounded initial datum β_0 (with suitable boundary conditions). Given a spatial homogeneous cartesian grid $I_j = [x_{j-1/2}, x_{j+1/2}]$ with centers x_j and a time discretization $t^n = n\Delta t$, we are interested in numerical schemes reading:

$$\frac{\beta_j^{n+1} - \beta_j^n}{\Delta t} + u \frac{\beta_{j+1/2}^n - \beta_{j-1/2}^n}{\Delta x} = 0. \tag{2}$$

This scheme is determined by choosing numerical fluxes $\beta_{j+1/2}^n$. To ensure L^∞ stability and the TVD property, we would like the fluxes to be such that:

$$b_j^n := \min\left(\beta_j^n, \beta_{j-1}^n\right) \leq \beta_j^{n+1} \leq B_j^n := \max\left(\beta_j^n, \beta_{j-1}^n\right). \tag{3a}$$

A sufficient condition to guarantee (3a) is:

$$\beta_{j+1/2}^n \in \mathcal{B}_j^n := \left[b_{j+1}^n, B_{j+1}^n\right] \cap \left[B_j^n + \frac{\Delta x}{u\Delta t}\left(\beta_j^n - B_j^n\right), b_j^n + \frac{\Delta x}{u\Delta t}\left(\beta_j^n - b_j^n\right)\right]. \tag{3b}$$

As the problem consists in handling discontinuous functions, we would like to avoid any numerical diffusion. However, this phenomenon is necessary so that the scheme may be stable. Després and Lagoutière's idea was to combine advantages from both upwind and downwind strategies by choosing the numerical fluxes as close as possible to the downwind flux (the downwind scheme is unconditionally unstable but nondiffusive), which lead them to set:

$$\beta_{j+1/2}^n = \underset{v \in \mathcal{B}_j^n}{\operatorname{argmin}} \left|\beta_{j+1}^n - v\right|. \tag{4}$$

The strategy (2)–(4) under the CFL condition:

$$\frac{u\Delta t}{\Delta x} \leq 1 \tag{5}$$

is called the *antidiffusive* scheme.

1. Prove that if Condition (3b) is satisfied, so is (3a).
2. Show that the upwind scheme may be written as (2) with fluxes satisfying (3b). Deduce that $\mathcal{B}_j^n$ is nonempty.
3. Solve the optimization problem (4).
4. Implement the antidiffusive scheme for $\beta_0 = \mathbf{1}_{[0,0.5]}$ in the domain $[0,1]$ with periodic conditions.
5. Carry out simulations in order to compare results with other schemes and show the order of the method.

2 1D Euler equations

We consider in this part the compressible Euler equations in one space dimension:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x(\rho E u + pu) = 0. \end{cases} \tag{6}$$

The system is closed with the equation of state:

$$p = \rho(\gamma - 1)\left(E - \frac{u^2}{2}\right).$$

The algorithm is based on the Lagrange-projection method:[1] it is split into two steps, the first one concerning the evolution of variables in Lagrange coordinates, the second one the projection on the eulerian grid (which is equivalent to the resolution of an advection equation). Eqs. (6) are equivalent (under smoothness assumptions) to the system:

$$\rho D_t \boldsymbol{f} + \partial_x \boldsymbol{\mathcal{F}} = 0, \qquad \boldsymbol{f} = \begin{pmatrix} \tau \\ u \\ E \end{pmatrix} \text{ and } \boldsymbol{\mathcal{F}} = \begin{pmatrix} -u \\ p \\ pu \end{pmatrix},$$

where $D_t = \partial_t + u\partial_x$ and $\tau = \rho^{-1}$. The Lagrangian step reads:

$$\rho_i^n \frac{\boldsymbol{f}_i^* - \boldsymbol{f}_i^n}{\Delta t} + \frac{\boldsymbol{\mathcal{F}}_{i+1/2}^n - \boldsymbol{\mathcal{F}}_{i-1/2}^n}{\Delta x} = 0. \tag{7a}$$

The other step formally corresponds to the resolution of:

$$\partial_t(\rho \tilde{\boldsymbol{f}}) + \partial_x(u\rho \tilde{\boldsymbol{f}}) = 0, \qquad \tilde{\boldsymbol{f}} = \begin{pmatrix} 1 \\ u \\ E \end{pmatrix}.$$

[1] Which corresponds to an operator splitting.

Hence the projection scheme:

$$\frac{\rho_i^{n+1}\tilde{\boldsymbol{f}}_i^{n+1} - \rho_i^n \tilde{\boldsymbol{f}}_i^*}{\Delta t} + \frac{u_{i+1/2}^n \rho_{i+1/2}^* \tilde{\boldsymbol{f}}_{i+1/2}^* - u_{i-1/2}^n \rho_{i-1/2}^* \tilde{\boldsymbol{f}}_{i-1/2}^*}{\Delta x} = 0. \tag{7b}$$

Let us now focus on the definition of fluxes appearing in (7a) and (7b). For the first step, we set:

$$\begin{cases} p_{i+1/2}^n = \dfrac{p_i^n + p_{i+1}^n}{2} + \dfrac{(\rho c)_{i+1/2}^n}{2}(u_i^n - u_{i+1}^n), \\ u_{i+1/2}^n = \dfrac{u_i^n + u_{i+1}^n}{2} + \dfrac{1}{2(\rho c)_{i+1/2}^n}(p_i^n - p_{i+1}^n), \\ (\rho c)_{i+1/2}^n = \sqrt{\max\left\{\rho_i^n (c_i^n)^2, \rho_{i+1}^n (c_{i+1}^n)^2\right\} \min\left\{\rho_i^n, \rho_{i+1}^n\right\}}. \end{cases}$$

Here, c denotes the sound velocity given by $c = \sqrt{\gamma p/\rho}$. As for the second part, we follow the example of § 1 by introducing a parameter $\theta_{i+1/2} \in [0,1]$ such that:

$$\rho_{i+1/2}^* \tilde{\boldsymbol{f}}_{i+1/2}^* = \theta_{i+1/2} \rho_i^* \tilde{\boldsymbol{f}}_i^* + (1 - \theta_{i+1/2}) \rho_{i+1}^* \tilde{\boldsymbol{f}}_{i+1}^*.$$

The objective is to have $\theta_{i+1/2}$ as close as possible to 0 (for $u_{i+1/2}^n > 0$) or to 1 (otherwise) in order to avoid diffusion. Similarly to what we presented in the first part for the resolution of (1), the stability condition provides three intervals for $\theta_{i+1/2}$. It is possible to show that these three intervals are nonempty as well as their intersection (refer to Lagoutière's PhD. Thesis available on its web page).

1. Combining (7a) and (7b), show that the scheme is consistent with Eqs. (6).
2. Implement this method for the Sob shock tube in dimension 1.

3 Directions

You may write a report answering the questions and presenting relevant simulations to emphasize advantages and drawbacks of each method. You are expected to deliver codes and report on **March 8., 2011** by email to:

`thierry.goudon@inria.fr` ; `sebastian.minjeaud@inria.fr` ; `yohan.penel@inria.fr`

A presentation session will be held on **March 11., 2011 at 3pm**. You will outline the methods and present striking results in 20 minutes.