PW #2: *Burgers* equation

Given a scalar conservation law, there exist several schemes to solve the equation. But the solutions provided by these algorithms are not equivalent. For instance, it is relevant to compare accuracy on equivalent meshes, computational costs and time, stability or consistency conditions, and so on. This work is thus devoted to comparing some numerical schemes when solving the *Burgers* equation:

$$\partial_t u + \partial_x \mathscr{F}(u) = 0, \tag{1a}$$

with:

$$\mathscr{F}(u) = \frac{u^2}{2}. \tag{1b}$$

This problem is restricted to the **periodic boundary condition case** in the present work.

Let us introduce the homogeneous cartesian mesh $x_j = j\Delta x$ for a given mesh size Δx. The natural framework for solving Syst. (1) consists in defining cells of which x_j is the middle. Consequently, we note $x_{j+1/2} = \left(j + \frac{1}{2}\right)\Delta x$ such that x_j is the middle of $I_j =]x_{j-1/2}, x_{j+1/2}[$. Likewise, the time discretization is denoted by $t^n = n\Delta t$, $\Delta t > 0$. The approximation of the solution u at x_j and t^n is u_j^n (the unknowns are located in the center of the cells).

1 Non-conservative schemes

Provided that the solution u is smooth enough, Eq. (1a) may be rewritten as:

$$\partial_t u + u\partial_x u = 0. \tag{2}$$

On the one hand, given this new formulation, we can derive a natural scheme aiming at solving Eq. (2):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0, \qquad \text{if} \quad u_j^n \geq 0, \tag{3a}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0, \qquad \text{if} \quad u_j^n < 0. \tag{3b}$$

This scheme will be referred to as the **upwind non-conservative scheme**.

On the other hand, Eq. (2) may be considered as a transport equation for which the velocity field is unknown. The **method of characteristics** (MOC) provides a classical scheme for this kind of equations. It consists in tracking particules in their motion due to the velocity field (characteristic curves) in order to keep track of relevant values. More precisely, we introduce the characteristic flow as the solution to the ODE:

$$\frac{\mathrm{d}\mathscr{X}}{\mathrm{d}t} = u\big(t, \mathscr{X}(t; s, x)\big), \qquad \mathscr{X}(s; s, x) = x.$$

This flow represents the position at time t of a particule located in x at time s. Under smoothness assumptions on u, the existence (and uniqueness) of the solution $\mathscr{X}$ is guaranteed by the Cauchy-Lipschitz theorem. The fact still remains that in this case, u is unknown, as well as its smoothness. That is why we are only given a local existence.

The MOC relies on the following remark. If u is a solution to Syst. (1), then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u\big(t,\mathscr{X}(t;s,x)\big)\right] = 0.$$

This means that u is constant along characteristic curves:

$$u\big(t,\mathscr{X}(t;s,x)\big) = u(s,x).$$

This leads to the scheme (taking $t = t^n$, $s = t^{n+1}$ and $x = x_j$):

$$u_j^{n+1} = \hat{u}^n(\xi_j^n), \qquad \xi_j^n \approx \mathscr{X}(t^n;t^{n+1},x_j). \tag{4a}$$

This scheme hence consists in two steps:

- *Time step*: resolution of the ODE for $\mathscr{X}$ on $[t^n, t^{n+1}]$. A 1st-order approximation is:

$$\xi_j^n = x_j - u_j^n \Delta t. \tag{4b}$$

- *Space step*: after localizing ξ_j^n in the interval $[x_k, x_{k+1})$, this step is based on a spatial interpolation using values u_k^n and u_{k+1}^n. More precisely:

$$u_j^{n+1} = \frac{x_{k+1} - \xi_j^n}{\Delta x} u_{k+1}^n + \frac{\xi_j^n - x_k}{\Delta x} u_k^n. \tag{4c}$$

This scheme is nonlinear but unconditionally stable.

2 Conservative schemes

Among schemes solving Syst. (1), there exists a class called “conservative schemes” due to a L^1-norm conservation principle. They read:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+1/2}^n - F_{j-1/2}^n}{\Delta x} = 0, \tag{5a}$$

where $F_{j+1/2}^n$ is the numerical flux. This term comes from the finite volume framework. Indeed, when integrating Eq. (1a) over $[t^n, t^{n+1}] \times [x_{j-1/2}, x_{j+1/2}]$, we get:

$$U_j^{n+1} - U_j^n + \int_{t^n}^{t^{n+1}} \left[\mathscr{F}\big(u(t,x_{j+1/2})\big) - \mathscr{F}\big(u(t,x_{j-1/2})\big)\right] \mathrm{d}t = 0,$$

where the fluxes at the boundary of the cells appear. We get intersted in the sequel in numerical fluxes of the type:

$$F_{j+1/2}^n = \mathfrak{F}(u_j^n, u_{j+1}^n), \tag{5b}$$

for every flux $\mathfrak{F}$ satisfying the consistency condition $\mathfrak{F}(u,u) = \mathscr{F}(u)$.

Here are several possible fluxes:

- **Upwind conservative scheme**:

$$\mathfrak{F}_{upw}(u_j^n, u_{j+1}^n) = \begin{cases} \mathscr{F}(u_j^n) & \text{if } u_j^n \geq 0, \\ \mathscr{F}(u_{j+1}^n) & \text{if } u_j^n < 0. \end{cases}$$

- **Lax-Friedrichs**:

$$\mathfrak{F}_{LF}(u_j^n, u_{j+1}^n) = \frac{1}{2}\big(\mathscr{F}(u_j^n) + \mathscr{F}(u_{j+1}^n)\big) - \frac{\Delta t}{2\Delta x}\big(u_{j+1}^n - u_j^n\big).$$

- **Richtmyer-Lax-Wendroff**:

$$\mathfrak{F}_{LW}(u_j^n, u_{j+1}^n) = \mathscr{F}(u_{j+1/2}^{n+1/2}), \qquad \text{where } u_{j+1/2}^{n+1/2} = \frac{1}{2}\big(u_{j+1}^n + u_j^n\big) - \frac{\Delta t}{2\Delta x}\big(\mathscr{F}(u_{j+1}^n) - \mathscr{F}(u_j^n)\big).$$

- **MacCormack**:

$$\mathfrak{F}_{McC}(u_j^n, u_{j+1}^n) = \frac{1}{2}\left[\mathscr{F}(u_{j+1}^n) + \mathscr{F}(u_j^{n+1/2})\right], \qquad \text{where } u_j^{n+1/2} = u_j^n - \frac{\Delta t}{\Delta x}\big(\mathscr{F}(u_{j+1}^n) - \mathscr{F}(u_j^n)\big).$$

- **Godunov**: the idea of the Godunov scheme relies on a piecewise constant approximation over cells. More precisely, we consider at time t^n the function $\hat{u}^n$ such that $\hat{u}^n(t^n, x) = U_j^n$ if $x \in I_j$. U_j^n denotes the mean value of the approached solution $\hat{u}^n$ at time t^n over the cell I_j. To progress in time, we introduce Riemann problems at each interface $x_{j+1/2}$. Let w be the solution to the Riemann model problem:

$$\partial_t w + \partial_x \mathscr{F}(w) = 0, \qquad w(0, x; u_L, u_R) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0. \end{cases}$$

  Integrating Eq. (1a) over $[t^n, t^{n+1}] \times I_j$, the Godunov scheme writes as (5a) with:

$$\mathfrak{F}(u_j^n, u_{j+1}^n) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathscr{F}\big(\hat{u}^n(t, x_{j+1/2})\big)\,\mathrm{d}t = \mathscr{F}\big(\hat{u}^n(t^{n+1}, x_{j+1/2})\big),$$

  for $\hat{u}^n(t^{n+1}, x_{j+1/2}) = w(\Delta t, 0; U_j^n, U_{j+1}^n)$. For the *Burgers* equation, the solution to the Riemann problem is a similarity solution and reads $w(t, x; u_L, u_R) = \hat{w}\left(\frac{x}{t}; u_L, u_R\right)$ where:

$$\hat{w}(\xi; u_L, u_R) = \begin{cases} u_L & \text{if } \xi \leq u_L, \\ \xi & \text{if } u_L < \xi < u_R, \\ u_R & \text{if } \xi \geq u_R, \end{cases}$$

  if $u_L \leq u_R$ and:

$$\hat{w}(\xi; u_L, u_R) = \begin{cases} u_L & \text{if } \xi \leq \sigma, \\ u_R & \text{if } \xi > \sigma, \end{cases}$$

  if $u_L > u_R$ for $\sigma = \dfrac{u_L + u_R}{2}$.

3 Kinetic scheme

A scheme may be obtained by considering Syst. (1) as the limit of the following kinetic equation when $\varepsilon \to 0$:

$$\partial_t f + a(v)\partial_x f = \frac{1}{\varepsilon}\left(\chi_u(v) - f\right), \tag{6a}$$

with:

$$a(v) = \mathscr{F}'(v). \tag{6b}$$

The function $\chi_u(v)$ is defined by:

$$\chi_u(v) = \begin{cases} 1 & \text{if } 0 < v < u, \\ -1 & \text{if } u < v < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

and the unknows of Syst. (1) and Syst. (6) are linked by the relationship:

$$u(x,t) = \int_{\mathbb{R}} f(x,v,t)\,dv. \tag{8}$$

We use a spliting scheme to approach the solution of Syst. (6):

- **Step 1: Linear transport step**

$$\partial_t f + a(v)\partial_x f = 0. \tag{9}$$

  We consider here an upwind scheme:

$$f_j^{n+\frac{1}{2}}(v) = f_j^n(v) - \frac{\Delta t}{\Delta x}\left[a^+(v)\left(f_j^n(v) - f_{j-1}^n(v)\right) - a^-(v)\left(f_{j+1}^n(v) - f_j^n(v)\right)\right] \tag{10}$$

  where $a^+(v) = \max(a(v), 0)$ and $a^-(v) = \max(-a(v), 0)$.

- **Step 2: Collision step**

$$\partial_t f = \frac{1}{\varepsilon}\left(\chi_u - f\right). \tag{11}$$

  When $\varepsilon \to 0$, this step reduces to $f = \chi_u$ and thus, the second step of the scheme consists in solving:

$$f_j^{n+1}(v) = \chi_{u_j^{n+\frac{1}{2}}}(v), \tag{12}$$

  where $u_j^{n+\frac{1}{2}} = \int_{\mathbb{R}} f_j^{n+\frac{1}{2}}(v)\,dv$.

Noting that $u_j^{n+1} = \int_{\mathbb{R}} f_j^{n+1}(v)\,dv = \int_{\mathbb{R}} \chi_{u_j^{n+\frac{1}{2}}}(v)\,dv = u_j^{n+\frac{1}{2}}$, we can now return to variables of interest, that is u_j^n. Integrating Eq. (10) with respect to $v \in \mathbb{R}$ yields an equation of the form (5a) where the flux $F_{j+1/2}^n$ is defined by:

$$F_{j+1/2}^n = \int_{\mathbb{R}} a^+(v)\chi_{u_j^n} - a^-(v)\chi_{u_{j+1}^n}\,dv.$$

4 Directions

You may consider the periodic domain $[0,2]$ and three initial conditions for the *Burgers* equation:

- $u_1^0(x) = x$;
- $u_2^0(x) = \mathbf{1}_{[1/4,3/4]}(x)$;
- $u_3^0(x) = \exp\left(\frac{-(x-1/2)^2}{x(1-x)}\right)\mathbf{1}_{(0,1)}(x)$.

You may implement 1 non-conservative scheme, 3 conservative schemes and the kinetic scheme. In each case, try to highlight the **CFL condition**.
Eventually, compare the solutions to Syst. (1) and the ones to:

$$\partial_t v + \partial_x \hat{\mathscr{F}}(v) = 0,$$

with $v = u^2$ and $\hat{\mathscr{F}}(v) = \frac{2}{3} v^{3/2}$. These equations are equivalent in the smooth case.
You may deliver your results electronically at latest on **February 6, 2011**.

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