Dispersive models

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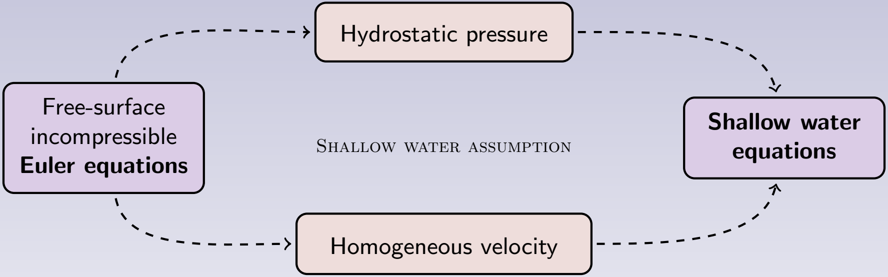
Inria evaluation seminar

March 13-14, 2018

Literature about free-surface flows

Free-surface incompressible **Euler equations**

Literature about free-surface flows



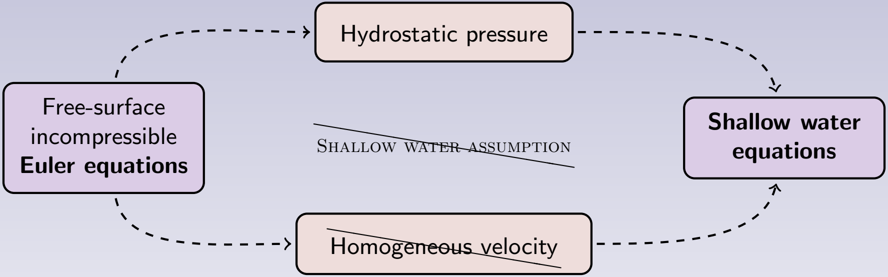
A. Barré de Saint-Venant, *Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leurs lits* (**C. R. Acad. Sci.** 73, 1871)

J.-F. Gerbeau, B. Perthame, *Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation* (**Discrete Contin. Dyn. Syst. Ser. B** 1(1), 2001)

S. Ferrari, F. Saleri, *A new two-dimensional Shallow Water model including pressure effects and slow varying bottom topography* (**Math. Model. Numer. Anal.** 38(2), 2004)

F. Marche, *Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects* (**Eur. J. Mech. B Fluids** 26(1), 2007)

Literature about free-surface flows



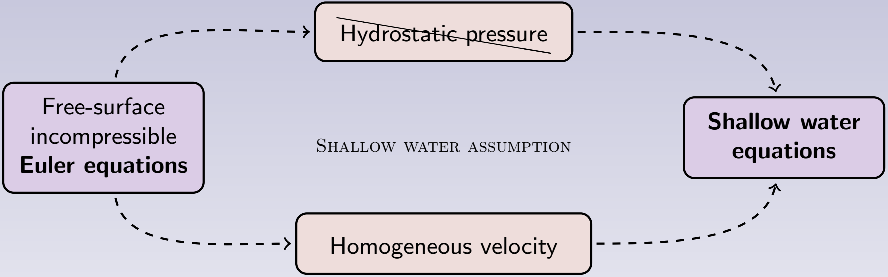
E. Audusse, M.-O. Bristeau, B. Perthame, J. Sainte-Marie, *A multilayer Saint-Venant system with mass exchanges for Shallow Water flows. Derivation and numerical validation* (**Math. Model. Numer. Anal.** 45(1), 2011)

F. Bouchut, V. Zeitlin, *A robust well-balanced scheme for multi-layer shallow water equations* (**Discrete Contin. Dyn. Syst. Ser. B** 13(4), 2010)

E.D. Fernández-Nieto, E.H. Koné, T. Morales de Luna, R. Bürger, *A multilayer shallow water system for polydisperse sedimentation* (**J. Comput. Phys.** 238, 2013)

Castro *et al.* '01 '04 '10, Narbona *et al.* '09 '13, . . .

Literature about free-surface flows



F. Serre, *Contribution à l'étude des écoulements permanents et variables dans les canaux* (**La Houille Blanche** 6, 1953)

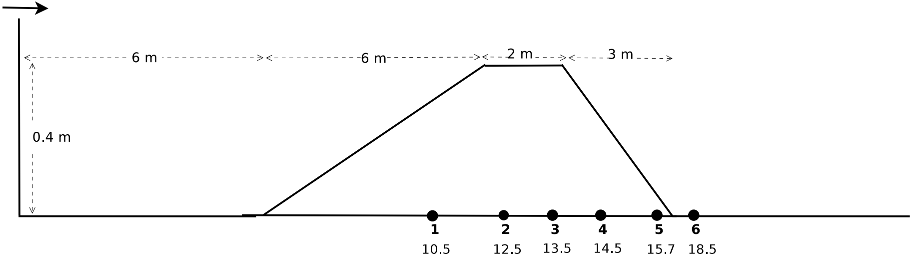
A.E. Green, P.M. Naghdi, *A derivation of equations for wave propagation in water of variable depth* (**J. Fluid Mech.** 78(2), 1976)

M.-O. Bristeau, J. Sainte-Marie, *Derivation of a non-hydrostatic shallow water model; Comparison with Saint-Venant and Boussinesq systems* (**Discrete Contin. Dyn. Syst. Ser. B** 10(4), 2008)

D. Lannes, P. Bonneton, *Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation* (**Phys. Fluids** 21(1), 2009)

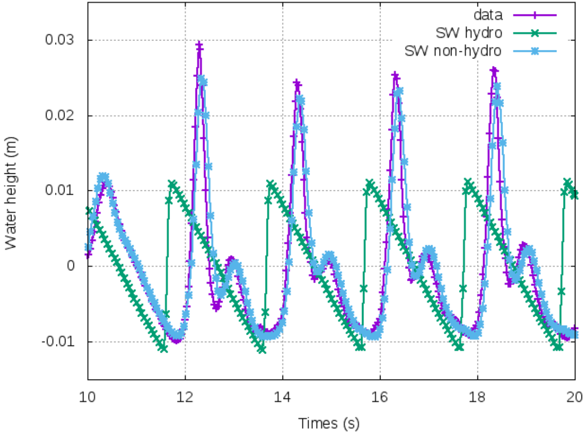
Peregrine '67, Madsen *et al.* '91 '96 '03 '06, Nwogu '93, Casulli *et al.* '95 '99, Yamazaki *et al.* '09, ...

Emphasis of non-hydrostatic effects



M.-W. Dingemans, *Wave propagation over uneven bottoms* (**Adv. Ser. Ocean Eng.**, 1997)

Emphasis of non-hydrostatic effects



A depth-averaged Euler model

Formulation: hydrostatic ($g\frac{H^2}{2}$) and hydrodynamic (p_{nh}) pressure components

$$\frac{\partial H}{\partial t} + \nabla_{\mathbf{x}} \cdot (H\overline{\mathbf{u}}) = 0$$

$$\frac{\partial (H\overline{\mathbf{u}})}{\partial t} + \nabla_{\mathbf{x}} \cdot (H\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) + \nabla_{\mathbf{x}} \left[H \left(p_{nh} + \frac{gH}{2} \right) \right] = -2 \left(p_{nh} + \frac{gH}{2} \right) \nabla_{\mathbf{x}} z_b$$

$$\frac{\partial (H\overline{w})}{\partial t} + \nabla_{\mathbf{x}} \cdot (H\overline{w}\ \overline{\mathbf{u}}) = 2p_{nh}$$

$$-\nabla_{\mathbf{x}} \cdot (H\overline{\mathbf{u}}) + \overline{\mathbf{u}} \cdot \nabla_{\mathbf{x}}(H + 2z_b) = 2\overline{w}$$

2D: $\mathbf{u} = u$, $\mathbf{x} = x$

3D: $\mathbf{u} = (u, v)$, $\mathbf{x} = (x, y)$

Main papers published by ANGE members

M.-O. Bristeau, A. Mangeney, J. Sainte-Marie, N. Seguin, *An energy-consistent depth-averaged Euler system: derivation and properties* (**Discrete Contin. Dyn. Syst. Ser. B** 20(4), 2015)

M.-O. Bristeau, J. Sainte-Marie, *Derivation of a non-hydrostatic shallow water model; Comparison with Saint-Venant and Boussinesq systems* (**Discrete Contin. Dyn. Syst. Ser. B** 10(4), 2008)

A depth-averaged Euler model

Compact formulation

$$\frac{\partial H}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}}) = 0$$

$$\frac{\partial (H\overline{\mathbf{U}})}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}} \otimes \overline{\mathbf{U}}) + \overline{\nabla}\left(g\frac{H^2}{2}\right) + \nabla_{sw}\, p_{nh} + gH\overline{\nabla} z_b = 0$$

$$\mathrm{div}_{sw}\, \overline{\mathbf{U}} = 0$$

Notations

$$\overline{\mathbf{U}} = \begin{pmatrix} \overline{\mathbf{u}} \\ \overline{w} \end{pmatrix} \qquad \nabla_{sw}\, p = \begin{pmatrix} H\nabla_{\mathbf{x}} p + p\nabla_{\mathbf{x}}(H + 2z_b) \\ -2p \end{pmatrix}$$

$$\overline{\nabla} = \begin{pmatrix} \nabla_{\mathbf{x}} \\ 0 \end{pmatrix} \qquad \mathrm{div}_{sw}\, \overline{\mathbf{U}} = \nabla_{\mathbf{x}} \cdot (H\overline{\mathbf{u}}) - \overline{\mathbf{u}} \cdot \nabla_{\mathbf{x}}(H + 2z_b) + 2\overline{w}$$

A depth-averaged Euler model

Compact formulation

$$\frac{\partial H}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}}) = 0$$

$$\frac{\partial (H\overline{\mathbf{U}})}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}} \otimes \overline{\mathbf{U}}) + \overline{\nabla}\left(g\frac{H^2}{2}\right) + \nabla_{sw}\, p_{nh} + gH\overline{\nabla} z_b = 0$$

$$\mathrm{div}_{sw}\, \overline{\mathbf{U}} = 0$$

Numerical strategy

Time splitting (projection/correction): Hyperbolic solver / Dispersive solver

A depth-averaged Euler model

Compact formulation

$$\frac{\partial H}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}}) = 0$$

$$\frac{\partial (H\overline{\mathbf{U}})}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}} \otimes \overline{\mathbf{U}}) + \overline{\nabla}\left(g\frac{H^2}{2}\right) + \nabla_{sw}\, p_{nh} + gH\overline{\nabla} z_b = 0$$

$$\mathrm{div}_{sw}\, \overline{\mathbf{U}} = 0$$

Numerical strategy

Time splitting (projection/correction): Hyperbolic solver / Dispersive solver

$$\int_\Omega \mathbf{V} \cdot \nabla_{sw}\, f \,\mathrm{d}\mathbf{x} = -\int_\Omega f \,\mathrm{div}_{sw}\, \mathbf{V} \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} Hf\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}\sigma$$

Poisson equation: $-\mathrm{div}_{sw}\left(\frac{1}{H}\nabla_{sw}\, p_{nh}\right) = 0$

Different possible variational formulations: inf-sup conditions satisfied

A depth-averaged Euler model

Compact formulation

$$\frac{\partial H}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}}) = 0$$

$$\frac{\partial (H\overline{\mathbf{U}})}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}} \otimes \overline{\mathbf{U}}) + \overline{\nabla}\left(g\frac{H^2}{2}\right) + \nabla_{sw}\, p_{nh} + gH\overline{\nabla} z_b = 0$$

$$\mathrm{div}_{sw}\, \overline{\mathbf{U}} = 0$$

Properties of the FV/FE strategy

- Positivity of H under a suitable CFL condition
- Preservation of equilibrium states (lake-at-rest)
- Semi-discrete entropy inequality

A depth-averaged Euler model

Compact formulation

$$\frac{\partial H}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}}) = 0$$

$$\frac{\partial (H\overline{\mathbf{U}})}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}} \otimes \overline{\mathbf{U}}) + \overline{\nabla}\left(g\frac{H^2}{2}\right) + \nabla_{sw}\, p_{nh} + gH\overline{\nabla} z_b = 0$$

$$\mathrm{div}_{sw}\, \overline{\mathbf{U}} = 0$$

Main papers published by ANGE members

- N. Aïssiouene, M.-O. Bristeau, E. Godlewski, J. Sainte-Marie, *A combined finite volume – finite element scheme for a dispersive shallow water system* (**Netw. Heterog. Media** 11(1), 2016)
- N. Aïssiouene, M.-O. Bristeau, E. Godlewski, A. Mangeney, C. Parés, J. Sainte-Marie, *A two-dimensional method for a dispersive shallow water model* (submitted)
- M. Parisot, *Entropy-satisfying scheme for a hierarchy of dispersive reduced models of free surface flow, Part I* (submitted)

A depth-averaged Euler model

Compact formulation

$$\frac{\partial H}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}}) = 0$$

$$\frac{\partial (H\overline{\mathbf{U}})}{\partial t} + \overline{\nabla} \cdot (H\overline{\mathbf{U}} \otimes \overline{\mathbf{U}}) + \overline{\nabla}\left(g\frac{H^2}{2}\right) + \nabla_{sw}\, p_{nh} + gH\overline{\nabla} z_b = 0$$

$$\mathrm{div}_{sw}\, \overline{\mathbf{U}} = 0$$

Open questions and barriers: Find the good balance between accuracy and efficiency

- Determine the most practical variational formulation
- Decrease the computational time: currently prohibitive for the 2D simulations of 3D flows

Elliptic part: change of formulation

From a mixed formulation on (p, u) ...

$$\begin{aligned} H u + \nabla_{sw}\, p &= g && \text{on } \Omega \\ \operatorname{div}_{sw} u &= f && \text{on } \Omega \end{aligned}$$

... to a conform formulation on p

$$\begin{aligned} -\operatorname{div}_{sw}\left(\frac{1}{H}\nabla_{sw}\, p\right) &= f - \operatorname{div}_{sw}\left(\frac{1}{H} g\right) && \text{on } \Omega \\ u &= \frac{1}{H}\left(g - \nabla_{sw}\, p\right) && \text{on } \Omega \end{aligned}$$

under assumption $0 < \underline{H} \leq H(\mathbf{x}) \leq \overline{H}$

Ani Miraçi's Master internship (summer 2017):

- conforming method: easier to implement & smaller linear system
- on simple 1D tests: similar accuracy as mixed formulation

Elliptic part: gradient discretization method (GDM)

- **Weak formulation :**

Find $p \in H_0^1(\Omega)$ such that $\forall \hat{p} \in H_0^1(\Omega)$,

$$\int_\Omega \frac{1}{H}(H\nabla p + p\nabla\zeta)\cdot(H\nabla\hat{p} + \hat{p}\nabla\zeta)\,\mathrm{d}x + \int_\Omega \frac{4p\hat{p}}{H}\,\mathrm{d}x = \int_\Omega \left(\widetilde{f}\hat{p} + \widetilde{g}\cdot\nabla\hat{p}\right)\,\mathrm{d}x$$

- **Gradient Discretization method :**

Find $p_{\mathcal{D}} \in X_{\mathcal{D}},0 \subset \mathbb{R}^{N_{\mathcal{D}}}$ such that $\forall \hat{p}_{\mathcal{D}} \in X_{\mathcal{D}},0$,

$$\int_\Omega \frac{1}{H}(H\nabla_{\mathcal{D}} p_{\mathcal{D}} + \Pi_{\mathcal{D}} p_{\mathcal{D}}\nabla\zeta)\cdot(H\nabla_{\mathcal{D}}\hat{p}_{\mathcal{D}} + \Pi_{\mathcal{D}}\hat{p}_{\mathcal{D}}\nabla\zeta)\,\mathrm{d}x + \int_\Omega \frac{4\Pi_{\mathcal{D}} p_{\mathcal{D}}\Pi_{\mathcal{D}}\hat{p}_{\mathcal{D}}}{H}\,\mathrm{d}x = \int_\Omega \left(\widetilde{f}\Pi_{\mathcal{D}}\hat{p}_{\mathcal{D}} + \widetilde{g}\cdot\nabla_{\mathcal{D}}\hat{p}_{\mathcal{D}}\right)\,\mathrm{d}x$$

Gradient Discretization (GD) : defined by the triplet $(X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$
3 properties *(coercivity, GD-consistency, limit conformity)* $\Rightarrow$ error estimate

Simple example: conforming $\mathbb{P}_1$ Finite Elements
On a triangular/tetrahedral mesh, $\mathcal{N}$ = set of nodes of the mesh

- $X_{\mathcal{D},0} = \{u = (u_N)_{N\in\mathcal{N}} \mid u_N = 0 \text{ if } N \in \partial\Omega\}$
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to C(\Omega) \qquad u \mapsto u_h = \sum_{N\in\mathcal{N}} u_N \phi_N$

  with ϕ_N $\mathbb{P}_1$ FE shape function associated to node N
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d \qquad u \mapsto \nabla_{\mathcal{D}} u = \nabla u_h$ *(piecewise constant function)*

GD method includes (e.g.):

- (non) conforming Galerkin methods
- mass lumping (with a suitable operator $\Pi_{\mathcal{D}}$)
- “finite volume style” methods (MPFA, SUSHI)

Ongoing work (Virgile Dubos' PhD thesis)

Resources

R. Eymard, C. Guichard, *Discontinuous Galerkin gradient discretisations for the approximation of second-order differential operators in divergence form* (**Comput. Appl. Math.**, 2017)

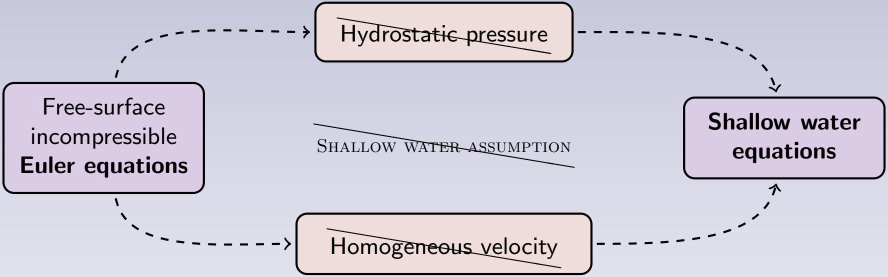
J. Droniou, R. Eymard, T. Gallouët, C. Guichard, R. Herbin, *The gradient discretisation method* (to appear in **Maths & Applications**)

J. Droniou, R. Eymard, R. Herbin, *Gradient schemes: generic tools for the numerical analysis of diffusion equations* (**M2AN** 50(3), 2016)

Plan

- Numerical analysis of the elliptic part thanks to GDM framework :
  1. on classical operator ∇, div
  2. on shallow-water operators ∇_{sw}, div_{sw}
- 2D numerical test using the conform formulation on a simplified coupled problem

Literature about free-surface flows

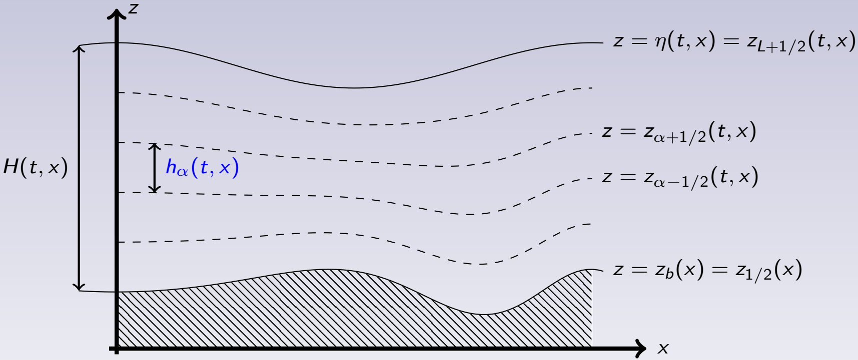


Derivation of multilayer non-hydrostatic models

International collaboration with Andalucía universities (Córdoba, Málaga, Sevilla) funded by French CNRS

E. Fernández-Nieto, M. Parisot, Y. Penel, J. Sainte-Marie, *A hierarchy of non-hydrostatic layer-averaged approximations of Euler equations for free surface flows* (**Commun. Math. Sci.**, to appear)

Multilayer framework



Height decomposition: $h_\alpha(t,x) = \ell_\alpha H(t,x)$ with $\ell_\alpha \in (0,1)$ and $\sum_{\alpha=1}^{L} \ell_\alpha = 1$

$\mathcal{L}_\alpha(t,x) = [z_{\alpha-1/2}(t,x), z_{\alpha+1/2}(t,x)]$

Hierarchy of layerwise-averaged models: LDNH

$u \in \mathbb{P}_0(\mathcal{L}_\alpha)$

Set $\overline{u} = \sum_{\alpha=1}^{L} \ell_\alpha u_\alpha$. Then the model reads

$$\partial_t H + \partial_x (H\overline{u}) = 0$$

$$\partial_t(h_\alpha u_\alpha) + \partial_x \left(h_\alpha u_\alpha^2 + h_\alpha q_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm})$$

$$\partial_t(h_\alpha w_\alpha) + \partial_x (h_\alpha u_\alpha w_\alpha) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

Hierarchy of layerwise-averaged models: LDNH_2

$u \in \mathbb{P}_0(\mathcal{L}_\alpha) \quad w \in \mathbb{P}_1(\mathcal{L}_\alpha) \quad q \in \mathbb{P}_2(\mathcal{L}_\alpha) \quad K = \frac{u^2+w^2}{2} \in \mathbb{P}_2(\mathcal{L}_\alpha)$

Set $\overline{u} = \sum_{\alpha=1}^{L} \ell_\alpha u_\alpha$. Then the model reads

$$\partial_t H + \partial_x (H\overline{u}) = 0$$

$$\partial_t(h_\alpha u_\alpha) + \partial_x \left(h_\alpha u_\alpha^2 + h_\alpha q_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm})$$

$$\partial_t(h_\alpha w_\alpha) + \partial_x (h_\alpha u_\alpha w_\alpha) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

$$\frac{\partial_t(h_\alpha\sigma_\alpha) + \partial_x(h_\alpha\sigma_\alpha u_\alpha)}{2\sqrt{3}} + \mathcal{Q}_{\alpha+1/2} - \mathcal{Q}_{\alpha-1/2} = q_\alpha - \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2}$$

$$w_\alpha - u_\alpha \partial_x z_\alpha + \sum_{\beta=1}^{\alpha-1} \partial_x(h_\beta u_\beta) + \frac{1}{2}\partial_x(h_\alpha u_\alpha) = 0$$

Hierarchy of layerwise-averaged models: LDNH_1

$u \in \mathbb{P}_0(\mathcal{L}_\alpha) \quad w \in \mathbb{P}_1(\mathcal{L}_\alpha) \quad q \in \mathbb{P}_2(\mathcal{L}_\alpha) \quad K = \frac{u^2+w^2}{2} \in \mathbb{P}_0(\mathcal{L}_\alpha)$

Set $\overline{u} = \sum_{\alpha=1}^{L} \ell_\alpha u_\alpha$. Then the model reads (consistent only for $\ell_\alpha = \frac{1}{L}$)

$$\partial_t H + \partial_x \left(H\overline{u}\right) = 0$$

$$\partial_t(h_\alpha u_\alpha) + \partial_x \left(h_\alpha u_\alpha^2 + h_\alpha q_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm})$$

$$\partial_t(h_\alpha w_\alpha) + \partial_x \left(h_\alpha u_\alpha w_\alpha\right) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

$$\underbrace{\frac{\partial_t(h_\alpha \sigma_\alpha) + \partial_x(h_\alpha \sigma_\alpha u_\alpha)}{2\sqrt{3}}}_{\to 0} + \widetilde{\mathcal{Q}}_{\alpha+1/2} - \widetilde{\mathcal{Q}}_{\alpha-1/2} = q_\alpha - \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2}$$

$$w_\alpha - u_\alpha \partial_x z_\alpha + \sum_{\beta=1}^{\alpha-1} \partial_x(h_\beta u_\beta) + \frac{1}{2}\partial_x(h_\alpha u_\alpha) = 0$$

Hierarchy of layerwise-averaged models: LDNH_0

$u \in \mathbb{P}_0(\mathcal{L}_\alpha) \quad w \in \mathbb{P}_0(\mathcal{L}_\alpha) \quad q \in \mathbb{P}_1(\mathcal{L}_\alpha) \quad K = \frac{u^2+w^2}{2} \in \mathbb{P}_0(\mathcal{L}_\alpha)$

Set $\overline{u} = \sum_{\alpha=1}^{L} \ell_\alpha u_\alpha$. Then the model reads

$$\partial_t H + \partial_x \left(H\overline{u}\right) = 0$$

$$\partial_t(h_\alpha u_\alpha) + \partial_x \left(h_\alpha u_\alpha^2 + h_\alpha q_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm})$$

$$\partial_t(h_\alpha w_\alpha) + \partial_x \left(h_\alpha u_\alpha w_\alpha\right) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

$$\underbrace{\frac{\partial_t(h_\alpha \sigma_\alpha) + \partial_x(h_\alpha \sigma_\alpha u_\alpha)}{2\sqrt{3}}}_{\to 0} + \underbrace{\mathcal{Q}_{\alpha+1/2} - \mathcal{Q}_{\alpha-1/2}}_{\to 0} = q_\alpha - \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2}$$

$$w_\alpha - u_\alpha \partial_x z_\alpha + \sum_{\beta=1}^{\alpha-1} \partial_x(h_\beta u_\beta) + \frac{1}{2}\partial_x(h_\alpha u_\alpha) = 0$$

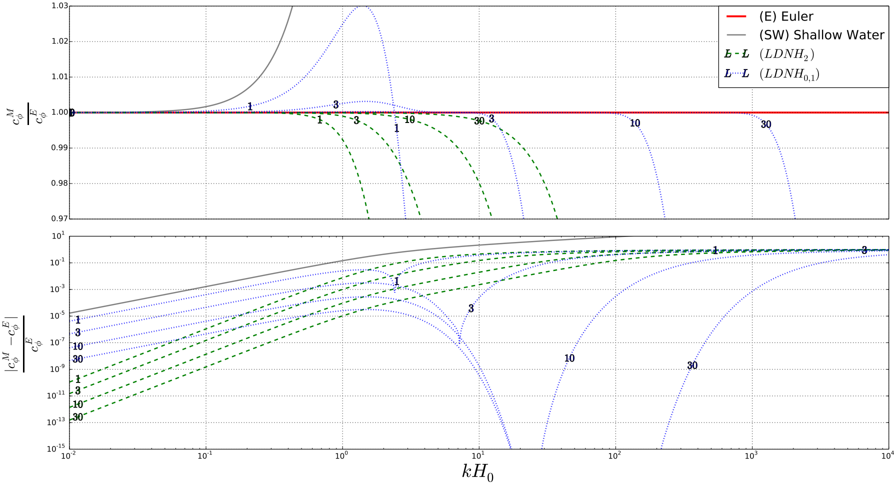
Properties

- Energy estimate:

$$\partial_t \left(\sum_{\alpha=1}^{L} h_\alpha \left(K_\alpha + g z_\alpha + p^{atm} \right) \right) + \partial_x \left(\sum_{\alpha=1}^{L} h_\alpha u_\alpha \left(K_\alpha + q_\alpha + g\eta + p^{atm} \right) \right) \leq H \partial_t p^{atm} + (gH + q_{1/2}) \partial_t z_b$$

- Conservation of the global volume
- Explicit dispersion relation no matter what the number of layers (linearization aroud the lake-at-rest)
- Convergence of the dispersion relation to the Airy formula when $L \to +\infty$

Linear dispersion relation



Conclusions

- First simulations: strong impact of the taking into account of the non-hydrostatic effects for applications of interest
- Parametrized hierarchy of models
- Qualitative results that show the relevance of the models
- Improvements required to go further from the numerical point of view

Coercivity, GD-consistency, Limit-conformity

$\mathcal{D} = (X_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ GD, $(\mathcal{D}_m)_{m\in\mathbb{N}}$ sequence of GDs

(P1) Coercivity

$$C_{\mathcal{D}} = \max_{v_{\mathcal{D}}\in X_{\mathcal{D},0}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Omega)^d}}$$

$(C_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded (discrete Poincaré inequality)

(P2) GD-consistency ("interpolation error" in FE)

$$S_{\mathcal{D}}(\varphi) = \min_{v_{\mathcal{D}}\in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v_{\mathcal{D}} - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v_{\mathcal{D}} - \nabla\varphi\|_{L^2(\Omega)^d} \right)$$

For all $\varphi \in H_0^1(\Omega)$, $S_{\mathcal{D}_m}(\varphi) \to 0$ as $m \to \infty$.

(P3) Limit-conformity

$$W_{\mathcal{D}}(\psi) = \max_{v_{\mathcal{D}}\in X_{\mathcal{D},0}\setminus\{0\}} \frac{1}{\|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2(\Omega)^d}} \left| \int_\Omega \nabla_{\mathcal{D}} v_{\mathcal{D}} \cdot \psi + \Pi_{\mathcal{D}} v_{\mathcal{D}} \mathrm{div}\psi \right|$$

For all $\psi \in H_{\mathrm{div}}(\Omega)$, $W_{\mathcal{D}_m}(\psi) \to 0$ as $m \to \infty$

Linear dispersion relation

Let us linearize around the so-called lake-at-rest steady state $(H_0, 0, 0, 0)$.

Proposition

There exists a plane wave solution $\left(\hat{H}, \hat{u}_\alpha, \hat{w}_\alpha, \hat{q}_\alpha\right) e^{i(kx-\omega t)}$ *to the linearized* LDNH_k *system provided the following dispersion relation holds*

$$c_L^2(kH_0) = \frac{\omega^2}{k^2 g H_0} = \frac{\mathcal{P}_L(kH_0)}{\mathcal{Q}_L(kH_0)}$$

where $\mathcal{P}_L$ *and* $\mathcal{Q}_L$ *are explicit polynomials. Moreover when the number of layers* L *goes to infinity,* c_L^2 *tends to*

$$c_{Airy}^2(kH_0) = \frac{\tanh(kH_0)}{kH_0}.$$

LDNH_2 for $L \in \{1, 2, 3\}$

L	$\mathcal{P}_L$	$\mathcal{Q}_L$
1	1	$1 + \frac{x^2}{3}$
2	$1 + \frac{x^2}{12}$	$1 + \frac{5x^2}{12} + \frac{7x^4}{576}$
3	$1 + \frac{x^2}{9} + \frac{5x^4}{2916}$	$1 + \frac{4x^2}{9} + \frac{19x^4}{972} + \frac{13x^6}{78732}$

LDNH_1 for $L \in \{1, 2, 3\}$

L	$\mathcal{P}_L$	$\mathcal{Q}_L$
1	1	$1 + \frac{x^2}{4}$
2	$1 + \frac{x^2}{16}$	$1 + \frac{3x^2}{8} + \frac{x^4}{256}$
3	$1 + \frac{5x^2}{54} + \frac{x^4}{1296}$	$1 + \frac{5x^2}{12} + \frac{5x^4}{432} + \frac{x^6}{46656}$

$L \geq 4$ ($\lambda = 3$ for LDNH_2 and $\lambda = 2$ for $\mathrm{LDNH}_{1/0}$)

$$\mathcal{P}_L(x) = \frac{1}{L}\left[\left(1 - \frac{x^2}{2\lambda L^2}\right)^{L-1} + \xi_{L-4}\left(1 - \frac{x^2}{2\lambda L^2}\right)^2 - \xi_{L-3}\left(1 + \frac{2\lambda - 1}{2\lambda}\frac{x^2}{L^2}\right)\right]$$

$$\mathcal{Q}_L(x) = \left(1 - \frac{x^2}{2\lambda L^2}\right)^{L-1}\left(1 + \frac{\lambda - 1}{2\lambda}\frac{x^2}{L^2}\right) + \left(1 - \frac{x^2}{2\lambda L^2}\right)^2 \frac{x^2 \xi_{L-4}}{2L^2} - \left(3 + \frac{2\lambda - 3}{2\lambda}\frac{x^2}{L^2}\right)\frac{x^2 \xi_{L-3}}{2L^2}$$

$$\xi_k = \frac{L^2}{x^2}\left(1 - \frac{x^2}{2\lambda L^2}\right)^{k+2} + \Xi_e \sum_{0 \leq 2m \leq k} \binom{k}{2m}\left(1 + \frac{\lambda - 1}{2\lambda}\frac{x^2}{L^2}\right)^{k-2m} \frac{x^{2m-1}}{L^{2m-1}}\left(1 + \frac{\lambda - 2}{4\lambda}\frac{x^2}{L^2}\right)^m + \Xi_o \sum_{0 \leq 2m+1 \leq k} \binom{k}{2m+1}\left(1 + \frac{\lambda - 1}{2\lambda}\frac{x^2}{L^2}\right)^{k-2m-1} \frac{x^{2m+1}}{L^{2m+1}}\left(1 + \frac{\lambda - 2}{4\lambda}\frac{x^2}{L^2}\right)^m$$

where $\Xi_e = -1 + \frac{1-3\lambda}{\lambda}\frac{x^2}{L^2} + \frac{-1+6\lambda-4\lambda^2}{4\lambda^2}\frac{x^4}{L^4}$ and $\Xi_o = -\frac{5}{2} + \frac{5(1-\lambda)}{2\lambda}\frac{x^2}{L^2} + \frac{-\frac{5}{2}+5\lambda-2\lambda^2}{4\lambda^2}\frac{x^4}{L^4}$.