A Simplified Model for a Low Mach Number Diphasic Flow

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ICPDE 2010 - For the 60th Birthday of M. Chipot

Poitiers – February, 20th, 2010

Introduction

Goal

Modeling of the evolution of bubbles in a nuclear reactor at the scale of bubbles (DNS)

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Modeling of the evolution of bubbles in a nuclear reactor at the scale of bubbles (DNS)

↔ Navier-Stokes equations for a diphasic compressible immiscible flow

↓

Diphasic Low Mach Number system (DLMN)

↕

Abstract Bubble Vibration model (ABV)

↑ Theoretical study

↑ Numerical simulations

Mach Number supposed to be small (still compressible flow but without acoustic wave effects)

Simplification but keeping a similar mathematical structure

OUTLINE

1. Derivation of the model
2. Theoretical results
3. Interface computations

PARTIAL OUTLINE

1. **Derivation of the model**
2. Theoretical results
3. Interface computations

Compressible Diphasic Navier-Stokes System

Starting point

Diphasic Compressible Navier-Stokes

$$\begin{cases} \partial_t(\rho Y_1) + \nabla\cdot(\rho Y_1 \mathbf{u}) = 0\,, \\ \partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0\,, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = -\nabla P + \nabla\cdot\sigma + \rho\mathbf{g}\,, \\ \partial_t(\rho E) + \nabla\cdot(\rho\mathbf{u}E) = -\nabla\cdot(P\mathbf{u}) + \nabla\cdot(\kappa\nabla T) + \nabla\cdot(\sigma\mathbf{u}) + \rho\mathbf{g}\cdot\mathbf{u}\,, \end{cases}$$

together with transmission and boundary conditions.

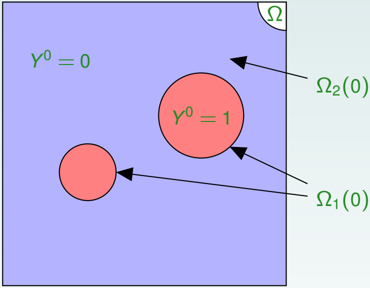
Nomenclature

- Y_1 : mass fraction of Fluid 1
- T : temperature
- P : pressure
- $\mathbf{u}$: global velocity
- E : total energy
- ρ : density
- $\mathbf{g}$: gravity field
- σ : Cauchy stress tensor
- κ : thermal conductivity
- $\theta = (Y_1, T, P)$

MODELING BUBBLES

Initial condition

$$Y_1(t=0,x) = Y^0(x) = \begin{cases} 1, & \text{if } x \in \Omega_1(0), \\ 0, & \text{if } x \in \Omega_2(0). \end{cases}$$



- Diphasic flow = non-miscible bi-fluid flow = Y_1 not regular
- Discontinuity of Y_1 = interface of the bubble
- $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$: open, bounded, smooth

The resolution of this equation with that initial condition amounts to determining for a domain $\Omega_1(t)$.

Hypotheses and Tools

Physics	Mathematics
Bounded domain (reactor)	$\Omega = [-1,1]^d$, $d \in \{2,3\}$
No void	$\rho > 0$
Linear elasticity	Linearized Cauchy tensor
Common physical properties for both fluids	Single nondimensioned system
Low Mach Number	Asymptotic expansion w.r.t. $\mathscr{M}_* \ll 1$
Existence of an entropy	$-T\mathrm{d}s = \mathrm{d}\varepsilon + P\mathrm{d}\tau$

Based on earlier Majda's & Embid's works on combustion ('84).

DLMN System

Diphasic Low Mach Number System: At order 0 in the asymptotic expansion, the system reads:

$$\begin{cases} \partial_t Y_1 + \mathbf{u} \cdot \nabla Y_1 = 0\,, \\ \nabla \cdot \mathbf{u} = G_\theta\,, \\ \rho(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla\Pi + 2\nabla\cdot[\mu D(\mathbf{u})] + \rho\mathbf{g}\,, \\ \rho c_p(\partial_t T + \mathbf{u}\cdot\nabla T) = \alpha T P'(t) + \nabla\cdot(\kappa\nabla T)\,, \\ P'(t) = H_\theta(t), \end{cases}$$

where P is the thermodynamic pressure, Π the dynamic pressure and:

$$G_\theta(t,x) := -\frac{D_t\rho}{\rho} = -\frac{1}{\Gamma}\frac{P'(t)}{P(t)} + \frac{\beta\nabla\cdot(\kappa\nabla T)}{P(t)},$$

$$H_\theta(t) := \frac{\displaystyle\int_\Omega \beta(\theta)\nabla\cdot\left[\kappa(\theta)\nabla T\right]\mathrm{d}\mathbf{x}}{\displaystyle\int_\Omega \frac{1}{\Gamma(\theta)}\,\mathrm{d}\mathbf{x}}.$$

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$$\begin{cases} \partial_t Y_1 + \mathbf{u}\cdot\nabla Y_1 = 0\,, \\ \nabla\cdot\mathbf{u} = G_\theta\,, \qquad \overset{\neq 0}{\Longrightarrow} \qquad \text{compressibility, elliptic contribution} \\ \rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla\Pi + 2\nabla\cdot[\mu D(\mathbf{u})] + \rho\mathbf{g}\,, \\ \rho c_p(\partial_t T + \mathbf{u}\cdot\nabla T) = \alpha T P'(t) + \nabla\cdot(\kappa\nabla T)\,, \\ P'(t) = H_\theta(t), \end{cases}$$

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DERIVED SYSTEMS FOR PRELIMARY STUDIES

Hodge Decomposition: $\mathbf{u} = \nabla\phi + \mathbf{w}$ with $\nabla \cdot \mathbf{w} = 0$ and boundary conditions.

Potential DLMN System $\mathbf{w} \to 0$

$$\begin{cases} \partial_t Y_1 + \nabla\phi \cdot \nabla Y_1 = 0\,, \\ \Delta\phi = G_\theta\,, \\ \rho c_p(\partial_t T + \nabla\phi \cdot \nabla T) = \alpha T P'(t) + \nabla \cdot (\kappa \nabla T)\,, \\ P'(t) = H_\theta(t). \end{cases}$$

Abstract Bubble Vibration Model $G_\theta \to G_Y$

$$\begin{cases} \partial_t Y_1 + \nabla\phi \cdot \nabla Y_1 = 0\,, \\ \Delta\phi(t,\mathbf{x}) = \psi(t) \left[Y_1(t,\mathbf{x}) - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega Y_1(t,\mathbf{y})\,\mathrm{d}\mathbf{y} \right]. \end{cases}$$

Partial Outline

1. Derivation of the model
2. Theoretical results
   - General study
   - 1D
3. Interface computations

Global issues

System of Partial Differential Equations

1. Properties of solutions
2. Existence of solutions depending on the regularity of Y^0
   - Strong solutions when $Y^0 \in \mathscr{H}^s$, s large enough
   - Weak solutions when $Y^0 \in L^\infty$
3. Uniqueness
4. Numerical simulations

Main tool: Energy estimates.

Set:

1. $\mathbb{Y} = \{Y \in L^\infty(\Omega) \ : \ Y(x) \in [0,1] \text{ for almost every } x \in \Omega\}$
2. $\mathscr{W}_{s,\mathscr{T}}(\Omega) = \mathscr{C}^0\left([0,\mathscr{T}], L^2(\Omega)\right) \cap L^\infty\left([0,\mathscr{T}], \mathscr{H}^s(\Omega)\right)$
3. $\mathscr{Z}_{\mathscr{T}}(\Omega) = L^\infty\left([0,\mathscr{T}], W^{1,\infty}(\Omega)\right)$

ALGEBRAIC PROPERTIES

$$
\begin{cases}
\partial_t Y + \nabla\phi \cdot \nabla Y = 0, \\
Y(0,\cdot) = Y^0, \\
\Delta\phi(t,\mathbf{x}) = \psi(t)\left(Y(t,\mathbf{x}) - |\Omega|^{-1}\displaystyle\int_\Omega Y(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}' \right), \\
\nabla\phi \cdot \mathbf{n}_{|\partial\Omega} = 0.
\end{cases}
$$

ALGEBRAIC PROPERTIES

Restrictions on the potential: ϕ_0 is prescribed by Y^0. In addition, we impose $\int_\Omega \phi(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$

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There is **at most** one weak solution in the class $\mathscr{X}_{\mathscr{T}}(\Omega)$

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If Ω is symmetric and Y^0 is even, then every solution belonging to $\mathscr{Z}_{\mathscr{T}}(\Omega)$ is **even**

ALGEBRAIC PROPERTIES

THEOREM (LAFITTE & DELLACHERIE, '05; PENEL, '10)

Assume $Y^0 \in \mathscr{H}^s(\Omega)$ **with** $s > \lfloor d/2 \rfloor + 1$ **and** $\psi \in \mathscr{C}^0([0,+\infty))$. **Then there exists** $\mathscr{T}_0 > 0$ **depending on** $\|Y^0\|_s$ **and** ψ **such that the ABV model has a unique classical solution** $Y_1 \in \mathscr{W}_{s,\mathscr{T}}(\Omega)$ **for some** $\mathscr{T}$ **at least greater than** $\mathscr{T}_0$.

$$\begin{cases} \partial_t Y + \nabla\phi \cdot \nabla Y = 0, \\ Y(0,\cdot) = Y^0, \\ \Delta\phi(t,\mathbf{x}) = \psi(t)\left(Y(t,\mathbf{x}) - |\Omega|^{-1} \int_\Omega Y(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}' \right), \\ \nabla\phi \cdot \mathbf{n}_{|\partial\Omega} = 0. \end{cases}$$

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LEMMA

Suppose that there exists a weak solution $Y_1(t,\mathbf{x}) = \mathbf{1}_{\Omega_1(t)}(\mathbf{x})$ where $\Omega_1(t) \subset \Omega$ and $\psi \in \mathscr{C}^0(0,+\infty)$, then the volume of the bubble is given by:

$$|\Omega_1(t)| = |\Omega| \frac{|\Omega_1(0)| \exp \int_0^t \psi(\tau)\mathrm{d}\tau}{|\Omega_2(0)| + |\Omega_1(0)| \exp \int_0^t \psi(\tau)\mathrm{d}\tau}.$$

PRELIMINARY RESULTS

1. Energy estimates for the transport equation: $\partial_t Y + \mathbf{u} \cdot \nabla Y = f$

$$\sup_{t\in[0,\mathscr{T}]} \|Y(t,\cdot)\|_r \leq e^{\chi_r(\mathscr{T})} \left(\|Y^0\|_r + \int_0^{\mathscr{T}} e^{-\chi_r(t)} \|f(t,\cdot)\|_r \,\mathrm{d}t \right),$$

with $\chi_r(t) = C_{adv,0}(r,d,\Omega) \int_0^t \|\nabla \mathbf{u}(\tau,\cdot)\|_{\max(s_0,r-1)} \,\mathrm{d}\tau.$

2. Elliptic regularity results for the Poisson equation

3. Embeddings $\mathscr{W}_{s,\mathscr{T}}(\Omega) \subset \mathscr{C}^0([0,\mathscr{T}],\mathscr{H}^{s'}) \subset \mathscr{C}^0([0,\mathscr{T}]\times\overline{\Omega})$ for any $s' < s$

4. Classical functional inequalities (Moser, interpolation, Gronwall, . . .)

Sketch of proof (1)

Iterative scheme:

1. $\Delta\phi^{(k)} = \psi(t)\left(Y^{(k)}(t,\mathbf{x}) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega Y^{(k)}(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right), \ \nabla\phi^{(k)}\cdot\mathbf{n}_{|\partial\Omega} = 0$
2. $\partial_t Y^{(k+1)} + \nabla\phi^{(k)}\cdot\nabla Y^{(k+1)} = 0, \ Y^{(k+1)}(0,\cdot) = Y^0$

Objectives:

1. to ensure the convergence of $\left(Y^{(k)}\right)$
2. to derive estimates in order to avoid a progressive loss of regularity as $k \to +\infty$
3. to check that the limit is a solution of the ABV model

Sketch of proof (2)

Proof of convergence:

1. Boundedness in $\mathscr{W}_{s,\mathscr{T}}(\Omega)$ which induces weak-$\star$ convergence to $\tilde{Y} \in \mathscr{W}_{s,\mathscr{T}}(\Omega)$ *via* the *Arzela-Ascoli* theorem and compactness arguments
2. Strong convergence to $Y \in \mathscr{W}_{0,\mathscr{T}}(\Omega)$
3. Finally, $Y = \tilde{Y} \in \mathscr{W}_{s,\mathscr{T}}(\Omega)$ and $Y^{(k)} \xrightarrow{\mathscr{W}_{s',\mathscr{T}}(\Omega)} Y$, for any $s' < s$

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Using energy estimates, we get:

$$\sup_{t\in[0,\mathscr{T}]} \|Y^{(k+1)}(t,\cdot)\|_s \leq \|Y^0\|_s \exp\left[C_{abv} \sup_{t\in[0,\mathscr{T}]} \|Y^{(k)}(t,\cdot)\|_s \int_0^{\mathscr{T}} |\psi(t)|\,\mathrm{d}t\right].$$

Since the sequence $u_{n+1} = u_0 e^{u_n}$ converges iff $u_0 \leq e^{-1}$, we show that:

$$\forall\, k \in \mathbb{N},\ \sup_{t\in[0,\mathscr{T}]} \|Y^{(k)}(t,\cdot)\|_s \leq e\|Y^0\|_s$$

under the hypothesis $\displaystyle\int_0^{\mathscr{T}} |\psi(t)|\,\mathrm{d}t \leq \frac{1}{e\cdot C_{abv}\cdot\|Y^0\|_s}$.

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On the other hand, we prove that:

$$e^{-\chi(t)} \left\| \left(Y^{(k+1)} - Y^{(k)} \right)(t,\cdot) \right\|_0 \leq C_{abv} \int_0^t e^{-\chi(\tau)} \left\| \left(Y^{(k)} - Y^{(k-1)} \right)(\tau,\cdot) \right\|_0 \mathrm{d}\tau$$

Iterating the process, we show that the series $\sum \left\| Y^{(k+1)} - Y^{(k)} \right\|_{0,\mathscr{T}}$ satisfies the Cauchy criterion in $\mathscr{W}_{0,\mathscr{T}}(\Omega)$ which is complete. Hence the strong convergence of $Y^{(k)}$ in $\mathscr{W}_{0,\mathscr{T}}(\Omega)$.

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We rewrite the iterative system under an intergal form and we conclude applying the dominated convergence theorem to show that the limit $(Y_1, \nabla\phi)$ is actually a solution.

TIME INTERVAL (1)

We proved the existence of a solution under the assumption:

$$e \cdot C_{abv} \cdot \|Y^0\|_s \int_0^{\mathscr{T}} |\psi(t)|\,\mathrm{d}t \leq 1.$$

1. We should bear in mind that this condition is sufficient and specific to the method used in the course of the proof: $\mathscr{T}$ is not necessarily optimal
2. Given an initial datum Y^0, we have a global existence for all $\psi \in L^1(0,+\infty)$ such that $\|\psi\|_{L^1} \leq \dfrac{1}{e \cdot C_{abv} \cdot \|Y^0\|_s}$
3. Given a pulse ψ and a time $\mathscr{T}$, there is a local existence for all initial data such that $\|Y^0\|_s \leq \dfrac{1}{e \cdot C_{abv} \cdot \|\psi\|_{L^1(0,\mathscr{T})}}$
4. If $\psi \equiv 0$, the solution is trivially $Y \equiv Y^0$ and $\mathscr{T} = +\infty$

TIME INTERVAL (2) – STILL IN PROGRESS

Continuation principle: if $Y_1(\mathscr{T},\cdot) \in \mathscr{H}^s$, we can apply the local existence theorem to the system:

$$\begin{cases} \partial_t Z + \nabla\varphi \cdot \nabla Z = 0, \\ Z(0,\cdot) = Y_1(\mathscr{T},\cdot), \\ \Delta\varphi(t,\mathbf{x}) = \tilde{\psi}(t)\left(Z(t,\mathbf{x}) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega Z(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right), \\ \nabla\varphi \cdot \mathbf{n}_{|\partial\Omega} = 0, \end{cases}$$

with $\tilde{\psi}(t) = \psi(\mathscr{T}+t)$. Thus, we can extend the solution Y_1 to a new time interval $[0,\mathscr{T}+\mathscr{T}_Z]$.

Let us denote $\mathscr{T}_k$ the time of existence at Step k in the continuation process. If the sequence is globally defined, there are two possibilities: either the series $\sum \mathscr{T}_k$ converges (and there is a local existence theorem), or it diverges (and we obtain a global solution).

REMARKS

In one space dimension, the Poisson equation is trivially solved and the ABV Model reads:

$$\partial_t Y(t,x) + \psi(t) \left(\int_{-L}^{x} Y(t,y)\,\mathrm{d}y - \frac{x+L}{2L} \int_{-L}^{L} Y(t,y)\,\mathrm{d}y \right) \partial_x Y(t,x) = 0.$$

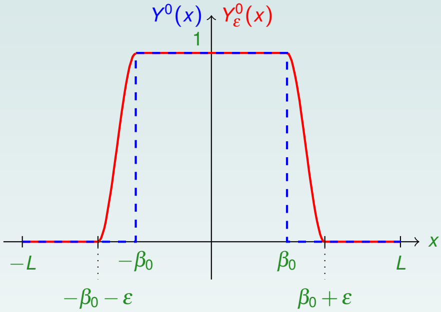
$\Longrightarrow$ continuity of the velocity field due to the embedding $\mathscr{H}^1 \subset \mathscr{C}^0$ in 1D.

Specificities of the 1D-case:

1. A single nonlinear integro-differential equation
2. Explicit solution in the irregular case
3. Finite propagation speed

Open problems: does the regularized solution converge to the explicit irregular solution?

EXPLICIT CALCULATIONS



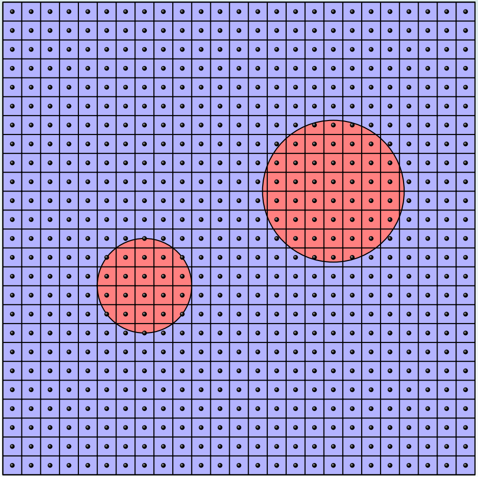
Explicit irregular solution in 1D

$$Y_1(t,x) = \mathbf{1}_{[-\beta(t),\beta(t)]}(x) \text{ with } \qquad \beta(t) = \frac{\beta_0}{\left(1-\frac{\beta_0}{L}\right)\exp\Psi(t)+\frac{\beta_0}{L}}.$$

Partial Outline

Discretization

Mismatch between the discrete and continuous levels



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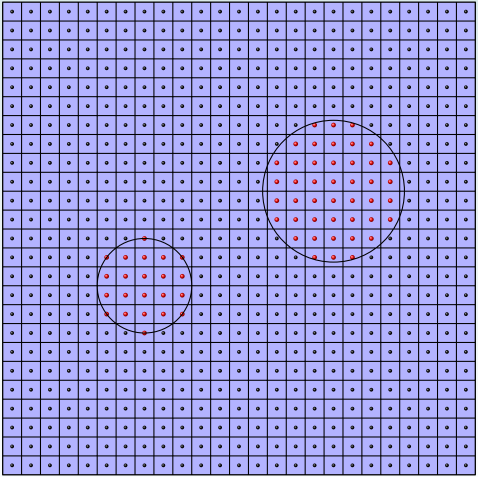


TEST ABV

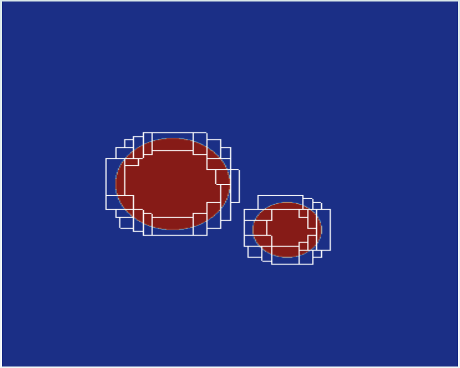
ABV model:

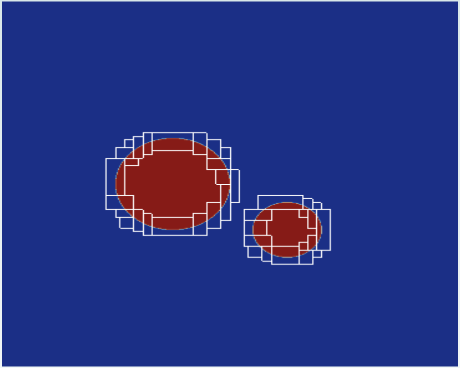
$$\begin{cases} \partial_t Y_1 + \nabla\phi \cdot \nabla Y_1 = 0, \\ \quad Y_1(0,\mathbf{x}) = Y^0(\mathbf{x}), \\ \Delta\phi = Y_1 - \frac{1}{4}\iint_{[-1,1]^2} Y_1(t,\mathbf{y})\,\mathrm{d}\mathbf{y}, \\ \quad \nabla\phi \cdot \mathbf{n}_{|\partial\Omega} = 0. \end{cases}$$

$\psi \equiv 1$: constant growth.

The initial datum Y^0 consists of two disjoint circles of different radii.

TEST ABV





100 $\times$ 100 grid with a refinement rate equal to 6

CONCLUSION

Done

- Existence and Uniqueness Theorem with an approximation of the time interval
- Study of the 1D-case
- Derivation of a numerical scheme to preserve interfaces

To do

- Approximating the time of existence for the DLMN system
- Enrichment with physical content
- Theoretical studies with irregular initial data


THANK YOU FOR YOUR ATTENTION